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A new expression for the form stress term in the vertically Lagrangian mean framework for the effect of surface waves on the upper ocean circulation

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Abstract

There is an ongoing discussion in the community concerning the wave-averaged momentum equations in the hybrid vertically Lagrangian and horizontally Eulerian (VL) framework and, in particular, the form stress term (representing the residual effect of pressure perturbations) which is thought to restrict the handling of higher order waves in terms of a perturbation expansion. The present study shows that the traditional pressure-based form stress term can be transformed into a set of terms that do not contain any pressure quantities but do contain the time derivative of a wave-induced velocity. This wave-induced velocity is referred to as the pseudomomentum in the VL framework, as it is analogous to the generalized pseudomomentum in Andrews and McIntyre. This enables the second expression for the wave-averaged momentum equations in the VL framework (this time for the development of the total transport velocity minus the VL pseudomomentum) to be derived together with the vortex force. The velocity-based expression of the form stress term also contains the residual effect of the turbulent viscosity, which is useful for understanding the dissipation of wave energy leading to transfer of momentum from waves to circulation. It is found that the concept of the virtual wave stress of Longuet-Higgins is applicable to quite general situations: it does not matter whether there is wind forcing or not, the waves can have slow variations, and the viscosity coefficient can vary in the vertical. These results provide a basis for revisiting the surface boundary condition used in numerical circulation models.

1. Introduction

Wave-averaged momentum equations for the effect of surface gravity waves on circulation in the upper ocean have been derived in either the three-dimensional Eulerian mean framework (e.g. McWilliams et al. 2004; Lane et al., 2007) or the three-dimensional Lagrangian mean framework (e.g. Pierson, 1962; Andrews and McIntyre, 1978, hereafter AM78; Ardhuin et al., 2008b). An ongoing discussion in recent studies is whether or not the corresponding equations can be obtained from the vertically Lagrangian and horizontally Eulerian (VL) mean framework that was introduced in prototype form by Mellor (2003) and Broström et al. (2008). Despite the discussion by Lane et al. (2007) and Ardhuin et al. (2008a), how to derive the wave-averaged momentum equations with the so-called vortex force "using the VL framework" remains an open question in the oceanographic community. On the other hand, as noted by Aiki and Greatbatch (2012, hereafter AG12), the VL framework offers a concise treatment of the viscous boundary condition at the sea surface, since the viscosity term of the wave-averaged momentum equations in the VL framework is written in a flux-divergence form. Thus, as shown in AG12, the VL framework can be used as an alternative to the three-dimensional Lagrangian framework of Pierson (1962) for explaining the virtual wave stress (VWS) of Longuet-Higgins (1953, 1960) [not to be confused with the radiation stress of Longuet-Higgins and Stewart (1964)]. To our knowledge the VWS has not been explained using the three-dimensional Lagrangian framework of AM78 in previous studies, apart from the attempt by Ardhuin et al. (2008b). The goal of the present study is to clarify the relationship between the VL framework and the three-dimensional Lagrangian framework. We show that the traditional pressure-based expression of the form stress term in the VL framework can be rewritten as a velocity-based expression, which we argue is the cornerstone for settling the discussion regarding both the vortex force in the VL framework and the relevance of the VWS to quite general situations.

The work of Lagrange (1788), who developed two different expressions for the momentum equations in Lagrangian coordinates, is highly relevant to the present study. Let $(x^\varepsilon, y^\varepsilon, z^\varepsilon)$ be the instantaneous position of a fluid particle in the Eulerian-Cartesian coordinates and $(u, v, w) \equiv (dx^\varepsilon/dt, dy^\varepsilon/dt, dz^\varepsilon/dt)$ be velocity where d/dt is the material derivative operator. Each fluid particle can be labelled by either its initial position (Lagrange, 1788; Lamb, 1932) or its low-pass filtered position (AM78), and is here symbolized as (a, b, c). The first expression (hereafter referred to as the direct expression) for the momentum equations in the three-dimensional Lagrangian coordinates reads,

$$\begin{pmatrix} \rho du/dt - Q^u \\ \rho dv/dt - Q^v \\ \rho dw/dt - Q^w \end{pmatrix} = - \begin{pmatrix} x_a^\varepsilon & y_a^\varepsilon & z_a^\varepsilon \\ x_b^\varepsilon & y_b^\varepsilon & z_b^\varepsilon \\ x_c^\varepsilon & y_c^\varepsilon & z_c^\varepsilon \end{pmatrix}^{-1} \begin{pmatrix} p_a \\ p_b \\ p_c \end{pmatrix}, \tag{1}$$

where ρ is density, (Q^u, Q^v, Q^w) is the sum of the effects of viscosity, gravitational acceleration, and the rotation of the Earth. In order to obtain an expression in the three-dimensional Lagrangian coordinates, the pressure gradient in the Eulerian coordinates $(p_{x^\varepsilon}, p_{y^\varepsilon}, p_{z^\varepsilon})$ has been rewritten using the pressure gradient in the three-dimensional Lagrangian coordinates (p_a, p_b, p_c) based on the standard chain-rule between partial differentials [see Eq. (C) on page 445 of Lagrange (1788)].[1] The Lagrangian average of (1) yields development equations for the Lagrangian mean (LM) velocity which is the sum of the Eulerian mean (EM) velocity and the Stokes-drift velocity (Stokes, 1847). Although the pressure gradient term of (1) is complicated, the forcing term is simpler than that in the second expression shown below, so that the direct expression has been often used in the studies of viscous surface waves [Chang, 1969; Ünlüata and Mei, 1970; Weber, 1983; Jenkins, 1987; Piedra-Cueva, 1995; Ng, 2004; all of these studies used Eq. (9) of Pierson (1962), see our Table 1].

[1]The chain rule is $\begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} = \begin{pmatrix} x_a^\varepsilon & y_a^\varepsilon & z_a^\varepsilon \\ x_b^\varepsilon & y_b^\varepsilon & z_b^\varepsilon \\ x_c^\varepsilon & y_c^\varepsilon & z_c^\varepsilon \end{pmatrix} \begin{pmatrix} \partial_{x^\varepsilon} \\ \partial_{y^\varepsilon} \\ \partial_{z^\varepsilon} \end{pmatrix}$.

The second expression (hereafter referred to as the transformed expression) for the momentum equations in the three-dimensional Lagrangian coordinates is obtained by multiplying (1) with the coordinate transformation matrix associated with the chain rule to yield [see Eq. (D) on page 446 of Lagrange (1788)],

$$\begin{pmatrix} x_a^\varepsilon & y_a^\varepsilon & z_a^\varepsilon \\ x_b^\varepsilon & y_b^\varepsilon & z_b^\varepsilon \\ x_c^\varepsilon & y_c^\varepsilon & z_c^\varepsilon \end{pmatrix} \begin{pmatrix} \rho du/dt - Q^u \\ \rho dv/dt - Q^v \\ \rho dw/dt - Q^w \end{pmatrix} = - \begin{pmatrix} p_a \\ p_b \\ p_c \end{pmatrix}. \tag{2}$$

Although the lhs is complicated, AM78 have presented a straightforward manipulation to render the Lagrangian average of (2) into development equations for the difference[2] between the LM velocity and the generalized pseudomomentum (or wave momentum, see Section 3 in AM78). Using this result of AM78, Leibovich (1980) has presented wave-averaged momentum equations that include the so-called vortex force,

$$(\partial_{t^\varepsilon} + \overline{\mathbf{V}}^\varepsilon \cdot \nabla^\varepsilon + \overline{w}^\varepsilon \partial_{z^\varepsilon}) \overline{\mathbf{V}}^\varepsilon = -\nabla^\varepsilon \mathcal{P} + \mathbf{V}^{Stokes} \times (\nabla^\varepsilon \times \overline{\mathbf{V}}^\varepsilon), \tag{3a}$$

$$(\partial_{t^\varepsilon} + \overline{\mathbf{V}}^\varepsilon \cdot \nabla^\varepsilon + \overline{w}^\varepsilon \partial_{z^\varepsilon}) \overline{w}^\varepsilon = -\partial_{z^\varepsilon} \mathcal{P} - g + \mathbf{V}^{Stokes} \cdot (\partial_{z^\varepsilon} \overline{\mathbf{V}}^\varepsilon - \nabla^\varepsilon \overline{w}^\varepsilon), \tag{3b}$$

where $\overline{\mathbf{V}}^\varepsilon$ and $\overline{w}^\varepsilon$ are the horizontal and vertical components of the EM velocity, respectively, $\mathbf{V}^{Stokes}$ is the horizontal component of the Stokes-drift velocity associated with surface gravity waves, ∂_{t^ε}, $\nabla^\varepsilon = (\partial_{x^\varepsilon}, \partial_{y^\varepsilon})$, and ∂_{z^ε} are the temporal, horizontal, and vertical gradient operators, respectively, in the Eulerian-Cartesian coordinates (see Table 2), g is gravitational acceleration, and $\mathcal{P}$ symbolizes the sum

[2] For irrotational wave motions in the vertical plane, the analytical expression of the generalized pseudomomentum of AM78 is identical to that of the Stokes-drift velocity at the leading order in terms of an asymptotic expansion, with a consequence that "the LM velocity minus the generalized pseudomomentum" approximates to the EM velocity. This is why (3a)-(3b) have been written for the development of the EM velocity. The LM velocity minus the generalized pseudomomentum has been referred to as the quasi-EM velocity in Jenkins (1989) and Ardhuin et al. (2008a,b).

of the Lagrangian average of nonhydrostatic pressure p and other scalar quantities (associated with the so-called Bernoulli head). Equations (3a)-(3b) omit the Coriolis term and the viscosity term for simplicity. The last term of each of (3a)-(3b) is the vortex force. The vortex force represents the interaction between an EM shear flow and the Stokes-drift flow associated with surface waves, and is appropriate to describe the maintenance of Langmuir Circulations (LCs, Langmuir, 1938). LCs play an important role in the vertical mixing of the surface mixed layer of the ocean (e.g. Skyllingstad et al., 1995; Polton and Belcher, 2007; Kukulka et al., 2010). Some prototypes of the vortex force have been derived by Craik and Leibovich (1976, hereafter CL76) using EM vorticity equations and by Garrett (1976) using EM depth-integrated momentum equations. Besides the theory of LCs, the vortex force has also been used in the modeling of the circulation in an inner coastal shelf region (e.g. McWilliams et al., 2004; Tang et al., 2007: Ardhuin et al., 2008b).

It can be said that the thickness-weighted-mean (TWM) momentum equations of Mellor (2003), Broström et al. (2008), and AG12 correspond to the Lagrangian average of (1) which is the direct expression of the Lagrangian momentum equations. This is because (i) the TWM momentum equations are written for the development of the TWM velocity whose role corresponds to that of the LM velocity in the three-dimensional Lagrangian framework and (ii) the TWM momentum equations contain the form stress term representing the residual effect of pressure perturbations. Nevertheless, in contrast to the Lagrangian average of (1), the TWM momentum equations contain the horizontal Reynolds stress term, which originates from the fact that the VL framework is Eulerian in the horizontal direction.

No previous study has attempted to derive the transformed expression for the TWM momentum equations corresponding to the Lagrangian average of (2). This is why, so far, the vortex force has not been obtained from the family of equations in the VL framework despite the discussion relating to this issue in the literature (cf. Jenkins and Ardhuin, 2004; Lane et al., 2007; Ardhuin et al., 2008a; Broström et al., 2008). A key step is the derivation of a new expression for the form stress written entirely using velocity variables, the subject of Section 2 of the present study. In Section 3 we show that the velocity-based expression of the form stress term contains the time derivative of a wave-induced velocity, resulting in the transformed expression for the TWM momentum equations, namely the expression rewritten for the development of the EM velocity, that contain the vortex force. We then introduce turbulent viscosity in Section 4 to show that the velocity-based expression of the form stress term also contains the residual effect of viscosity, and is useful for understanding the issue of how the dissipation of wave energy leads to the transfer of momentum from waves to circulation. Our finding is that the VWS of Longuet-Higgins (1953, 1960) is applicable in quite general situations and needs to be taken into account when considering the boundary conditions used in numerical circulation models. Section 5 presents a brief summary.

2. Mathematical development

We consider incompressible inviscid water of constant, uniform density in a non-rotating frame (rotation and viscosity are introduced in Section 4).

a. The thickness-weighted-mean equations of Aiki and Greatbatch (2012)

The incompressible condition and the momentum equations in the VL coordinates, (x, y, z, t), of AG12 are

$$(z^\varepsilon_z)_t + \nabla\cdot(z^\varepsilon_z\mathbf{V}) + (z^\varepsilon_z\varpi)_z = 0, \tag{4a}$$

$$(\partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z)z^\varepsilon = w, \tag{4b}$$

$$(\partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z)\mathbf{V} = -\nabla(p+\eta) + p_{z^\varepsilon}\nabla z^\varepsilon, \tag{4c}$$

$$(\partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z)w = -p_{z^\varepsilon}, \tag{4d}$$

where $z^\varepsilon = z^\varepsilon(x, y, z, t)$ is the instantaneous height of fluid particles in the standard Eulerian-Cartesian coordinates. The symbol ε, rather than c as used in AG12 and Aiki and Greatbatch (2013 - hereafter AG13), is used in the present study in order to preserve consistency with (1) and (2). In addition the symbol w^* in AG12 and AG13 has been replaced by the new symbol ϖ in the present study for convenience. The vertical coordinate, $z \equiv \overline{z^\varepsilon}$, is a low-pass filtered height coordinate and z_z^ε is the thickness. The horizontal coordinates x and y are the same as the Eulerian-Cartesian coordinates. The quantity $\mathbf{V} = (u, v)$ is the horizontal velocity vector, w is the vertical component of velocity, ϖ represents water flux through the surfaces of fixed z, $\nabla \equiv (\partial_x, \partial_y)$ is the lateral gradient operator along the surfaces of fixed z, and $\nabla z = 0$ is understood. The quantity p is the sum of the oceanic non-hydrostatic and atmospheric pressure and η is the instantaneous sea surface height. Table 2 presents a list of the symbols used in the text. All variables and quantities (such as $x, y, z, t, u, v, w, \varpi, z^\varepsilon, p$, and η) in the present manuscript have been non-dimensionalized, as in Appendix A of AG13. The non-dimensionalization is not essential but serves to simplify the mathematics.

The difference between the three-dimensional Lagrangian coordinates of AM78 and the VL coordinates is illustrated in Fig. 1. As the wave propagates rightward, the control cell of the three-dimensional Lagrangian coordinates (blue) rotates clockwise and returns to its original position. The movement of the control cell captures only high-frequency fluid motion (rather than the full motion of each fluid particle), which is why the control cell does not drift away despite the Stokes-drift and even though there could also be a background EM flow present in the horizontal and vertical directions. The control cell of the VL coordinates (red) moves like a piston whose thickness stretches and shrinks.

Momentum equations in a flux-divergence form can be obtained by multiplying each of (4c) and (4d) by the thickness z_z^ε and then using (4a) to give

$$(z_z^\varepsilon \mathbf{V})_t + \nabla\cdot(z_z^\varepsilon \mathbf{V}\mathbf{V}) + (z_z^\varepsilon \varpi \mathbf{V})_z = -z_z^\varepsilon \nabla(p+\eta) + p_z \nabla z^\varepsilon, \tag{5a}$$

$$(z_z^\varepsilon w)_t + \nabla\cdot(z_z^\varepsilon \mathbf{V} w) + (z_z^\varepsilon \varpi w)_z = -p_z. \tag{5b}$$

where $z_z^\varepsilon p_{z^\varepsilon} = p_z$ is understood. Low-pass temporal filtering each of (4a), (5a), and (5b) yields TWM equations[3] for incompressibility and the horizontal and vertical components of momentum,

$$\nabla\cdot\widehat{\mathbf{V}} + \widehat{\varpi}_z = 0, \tag{6a}$$

$$\widehat{\mathbf{V}}_t + \nabla\cdot(\widehat{\mathbf{V}}\widehat{\mathbf{V}}) + (\widehat{\varpi}\widehat{\mathbf{V}})_z + \mathcal{RS}^{\mathbf{V}} = -\nabla(\overline{p}+\overline{\eta}) + \mathcal{FS}^{\mathbf{V}}, \tag{6b}$$

$$\widehat{w}_t + \nabla\cdot(\widehat{\mathbf{V}}\widehat{w}) + (\widehat{\varpi}\widehat{w})_z + \mathcal{RS}^{w} = -\overline{p}_z, \tag{6c}$$

where $\overline{z_z^\varepsilon} \equiv 1$ (i.e. $\overline{z^\varepsilon} \equiv z$) has been used. The hat symbol is the TWM operator ($\widehat{A} \equiv \overline{z_z^\varepsilon A}$ for an arbitrary quantity A). The symbols $\mathcal{RS}^{\mathbf{V}}$ and $\mathcal{RS}^{w}$ in (6b)-(6c) are the Reynolds stress terms defined by

$$\mathcal{RS}^{A} \equiv \nabla\cdot(\overline{z_z^\varepsilon \mathbf{V}'' A''}) + (\overline{z_z^\varepsilon \varpi'' A''})_z, \tag{7}$$

for $A = u, v$, and w. The double-prime symbol is the deviation from the TWM ($A'' \equiv A - \widehat{A}$, compared at fixed z). The last term of (7) is given by ϖ'' (not w'') and thus is nearly zero. The fact that ϖ'' is nearly zero is attributed to the way the VL coordinates have been designed so that ϖ represents fluid motions associated with low-frequency fluid motions and not with the waves themselves. In particular AG12 have shown that $\varpi'' = \mathbf{V}''\cdot\nabla\overline{\eta}$ at the sea surface, which implies that the Reynolds stress vector $(\overline{z_z^\varepsilon \mathbf{V}'' A''}, \overline{z_z^\varepsilon \varpi'' A''})$ is aligned along the mean slope of the sea surface. The symbol $\mathcal{FS}^{\mathbf{V}}$ in (6b) is the

[3]The TWM equations originate from studies on mesoscale eddies (cf. Gallimore and Johnson, 1981; Andrews, 1983; Bleck, 1985; de Szoeke and Bennett, 1993; Greatbatch, 1998; Iwasaki, 2001; Greatbatch and McDougall, 2003; Aiki and Richards, 2008; Young, 2012).

form stress term defined by

$$\mathcal{FS}^{\mathbf{V}} \equiv -\overline{z_z'''\nabla(p'''+\eta''')} + \overline{p_z'''\nabla z'''}$$

$$= -[\overline{z'''\nabla(p'''+\eta''')}]_z + \nabla(\overline{z'''p_z'''}), \tag{8}$$

where the triple-prime symbol is the deviation from the unweighted mean ($A''' \equiv A - \overline{A}$, compared at fixed z). It should be noted that $z''' \equiv z^\varepsilon - \overline{z^\varepsilon} = z^\varepsilon - z$.

For low-pass filtered quantities, the VL coordinates (x, y, z, t) correspond to the standard Eulerian-Cartesian coordinates (Jacobson and Aiki, 2006). A nice feature of the total transport velocity $(\widehat{\mathbf{V}}, \widehat{\varpi})$ in (6a)-(6c) is that both the incompressible condition (6a) and the kinematic boundary condition ($\widehat{\varpi} = \overline{\eta}_t + \widehat{\mathbf{V}} \cdot \nabla\overline{\eta}$ at the sea surface, see AG12) are always satisfied without relying on an asymptotic expansion approach. The incompressibility condition allows us to rewrite the TWM momentum equations (6b)-(6c) in the form

$$\widehat{\mathcal{D}}_t\widehat{\mathbf{V}} + \mathcal{RS}^{\mathbf{V}} = -\nabla(\overline{p}+\overline{\eta}) + \mathcal{FS}^{\mathbf{V}}, \tag{9a}$$

$$\widehat{\mathcal{D}}_t\widehat{w} + \mathcal{RS}^{w} = -\overline{p}_z, \tag{9b}$$

where $\widehat{\mathcal{D}}_t \equiv \partial_t + \widehat{\mathbf{V}} \cdot \nabla + \widehat{\varpi}\partial_z$ is the material derivative operator based on the total transport velocity. The quantities $\widehat{w}$ and $\widehat{\varpi}$ are not the same mathematically but the difference is negligible as far as the present study is concerned, as is demonstrated in Section 3.

The wave-induced velocity in Mellor (2003) can be called the quasi-Stokes velocity following McDougall and McIntosh (2001). The quasi-Stokes velocity $(\mathbf{V}^{qs}, w^{qs})$ refers to the difference between the total transport velocity $(\widehat{\mathbf{V}}, \widehat{\varpi})$ and the EM velocity $(\overline{\mathbf{V}}^{\varepsilon}, \overline{w}^{\varepsilon})$,

$$\mathbf{V}^{qs} \equiv \widehat{\mathbf{V}} - \overline{\mathbf{V}}^{\varepsilon}(= \overline{\mathbf{V}} + \overline{z_z'''\mathbf{V}'''} - \overline{\mathbf{V}}^{\varepsilon}), \tag{10a}$$

$$w^{qs} \equiv \widehat{\varpi} - \overline{w}^{\varepsilon}(= \overline{w} - \overline{\mathbf{V}''' \cdot \nabla z'''} - \overline{w}^{\varepsilon}), \tag{10b}$$

where (4b) has been used.[4] The expressions in brackets will prove useful in Section 3. As noted in AG12, the quasi-Stokes velocity and the Stokes drift are closely related.

b. *A new expression of the form stress term*

The form stress term in (8) is based on the pressure fluctuation p''' so that it is not useful for some analytical treatments (Ardhuin et al., 2008a,b). Our view is that writing the form stress term in the TWM momentum equation (6b) in terms of pressure (i.e. as in (8)) implies a wave-averaged equation which corresponds to the Lagrangian average of equation (1): the direct expression for the momentum equation in the three-dimensional Lagrangian coordinates with the complicated pressure gradient term. We now show that the pressure-based form stress term in (8) can be transformed into a new expression where the pressure fluctuation p''' does not appear and which, in turn, provides a link to the Lagrangian average of the transformed expression for the Lagrangian momentum equation (2).

Substitution of $z^\varepsilon = z + z'''$ and $\mathcal{D}_t \equiv \partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z$ to (4a)-(4d) yields,

$$\mathcal{D}_t\mathbf{V} = -\nabla(p+\eta) - \underbrace{(\mathcal{D}_t w)}_{-p_{z^\varepsilon}}\nabla z''', \tag{11a}$$

$$\mathcal{D}_t w = -p_z - (\mathcal{D}_t w) z'''_z, \tag{11b}$$

Equation (11b) has been derived by multiplying (4d) with $1+z'''_z$ and noting that $(1+z'''_z)p_{z^\varepsilon} = z^\varepsilon_z p_{z^\varepsilon} = p_z$.

[4] Appendix D of AG13 provides an approximate expression for the EM quantities based on a Taylor expansion in the vertical direction with a continuous treatment of the vertical profile (i.e. no singular treatment of the vicinity of the sea surface), and is free from the traditional issue of how to handle regions above the surface troughs.

Using (11a)-(11b), we rewrite the form stress term (8) to give

$$\begin{aligned}
\mathcal{FS}^{\mathbf{V}} &= -\overline{z_z'''\nabla(p'''+\eta''')} + \overline{(\nabla z''')p_z'''} \\
&= -\overline{z_z'''\nabla(p+\eta)} + \overline{(\nabla z''')p_z} \\
&= \overline{z_z'''(\mathcal{D}_t\mathbf{V} + (\mathcal{D}_t w)\nabla z''')} - \overline{\nabla z'''(1+z_z''')(\mathcal{D}_t w)} \\
&= \overline{z_z'''(\mathcal{D}_t\mathbf{V})} - \overline{\nabla z'''(\mathcal{D}_t w)} \\
&= \overline{z_z'''(\mathcal{D}_t\mathbf{V})'''} - \overline{\nabla z'''(\mathcal{D}_t w)'''}, \qquad (12)
\end{aligned}$$

where no p''' appears at the last line. This expression has not, to our knowledge, been shown before the present study[5] and is the cornerstone of the present study. Note that (12) has been derived from a nonlinear equation system and thus is applicable to finite-amplitude waves including the Doppler effect by mean flows.

c. Manipulation of the velocity-based form stress term

In order to expand the velocity-based form stress term (12), we need to derive variants of (4a)-(4b). Using $z_z^\varepsilon = 1 + z_z'''$ and $A'' \equiv A - \widehat{A}$ for an arbitrary quantity, (4a) may be written

$$\begin{aligned}
\mathcal{D}_t z_z''' &= -z_z^\varepsilon[\nabla\cdot(\widehat{\mathbf{V}} + \mathbf{V}'') + (\widehat{\varpi} + \varpi'')_z] \\
&= -z_z^\varepsilon(\nabla\cdot\mathbf{V}'' + \varpi_z''), \qquad (13)
\end{aligned}$$

[5]Apart from the simplified version shown by AG13 at their Eq. (26).

where (6a) has been used to derive the second line. The material derivative operator may be expanded as,

$$\begin{aligned}\mathcal{D}_t &\equiv \partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z \\ &= \partial_t + (\widehat{\mathbf{V}} + \mathbf{V}'')\cdot\nabla + (\widehat{\varpi} + \varpi'')\partial_z \\ &= \widehat{\mathcal{D}}_t + \mathbf{V}''\cdot\nabla + \varpi''\partial_z. \end{aligned}\tag{14}$$

Substitution of (14) to (13) yields

$$\widehat{\mathcal{D}}_t z_z''' = -\nabla\cdot(z_z^\varepsilon\mathbf{V}'') - (z_z^\varepsilon\varpi'')_z.\tag{15}$$

On the other hand, (4b) may be written using (14) and $z^\varepsilon = z + z'''$, as

$$\begin{aligned} w &= (\partial_t + \mathbf{V}\cdot\nabla + \varpi\partial_z)z^\varepsilon \\ &= \widehat{\mathcal{D}}_t z^\varepsilon + (\mathbf{V}''\cdot\nabla + \varpi''\partial_z)z^\varepsilon \\ &= \widehat{\mathcal{D}}_t z''' + \widehat{\varpi} + \mathbf{V}''\cdot\nabla z''' + \varpi'' z_z^\varepsilon. \end{aligned}\tag{16}$$

The unweighted average of (16) yields

$$\overline{w} = \widehat{\varpi} + \overline{\mathbf{V}''\cdot\nabla z'''}.\tag{17}$$

Combining (17) and (16) then yields

$$w''' = \widehat{\mathcal{D}}_t z''' + (\mathbf{V}''\cdot\nabla z''')''' + \varpi'' z_z^\varepsilon.\tag{18}$$

The quantity ϖ'' is sufficiently small (as we shall see later), that all terms containing ϖ'' in the above are neglected in what follows.

We are now ready to expand the terms that make up the last line of (12). Using (14) and (15), the first term on the last line of (12), letting A be either u or v, can be written

$$
\begin{aligned}
\overline{z_z'''(\mathcal{D}_t A)'''} &= \overline{z_z'''(\mathcal{D}_t A)} \\
&= \overline{z_z'''(\widehat{\mathcal{D}}_t A + \mathbf{V}'' \cdot \nabla A)} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A}) - \overline{(\widehat{\mathcal{D}}_t z_z''')A} + \overline{z_z''' \mathbf{V}'' \cdot \nabla A} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A}) + \overline{[\nabla \cdot (z_z^\varepsilon \mathbf{V}'')]A} + \overline{z_z''' \mathbf{V}'' \cdot \nabla A} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A}) + \nabla \cdot (\overline{z_z^\varepsilon \mathbf{V}'' A}) - \overline{z_z^\varepsilon \mathbf{V}'' \cdot \nabla A} + \overline{z_z''' \mathbf{V}'' \cdot \nabla A} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A}) + \nabla \cdot (\overline{z_z^\varepsilon \mathbf{V}'' A}) - \overline{\mathbf{V}'' \cdot \nabla A} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A}) + \nabla \cdot (\overline{z_z^\varepsilon \mathbf{V}'' A}) - \overline{(\mathbf{V}''' - \overline{z_z''' \mathbf{V}'''}) \cdot \nabla A} \\
&= \widehat{\mathcal{D}}_t(\overline{z_z''' A'''}) + \mathcal{RS}^A - \overline{\mathbf{V}''' \cdot \nabla A'''} + \overline{z_z''' \mathbf{V}'''} \cdot \nabla \overline{A}, \qquad (19)
\end{aligned}
$$

where $\mathbf{V}'' \equiv \mathbf{V} - \widehat{\mathbf{V}} = (\overline{\mathbf{V}} + \mathbf{V}''') - \widehat{\mathbf{V}} = \mathbf{V}''' - \overline{z_z''' \mathbf{V}'''}$ has been used to derive the second last line, and $\nabla \cdot \overline{[z_z^\varepsilon \mathbf{V}''(\widehat{A} + A'')]} = \nabla \cdot (\overline{z_z^\varepsilon \mathbf{V}'' A''}) = \mathcal{RS}^A$ has been used to derive the last line.

Turning now to the second term on the last line of (12), we use (14) and (18), letting subscript X correspond to either ∂_x or ∂_y, to write

$$
\begin{aligned}
&-\overline{z_X'''(\mathcal{D}_t w)'''} = -\overline{z_X'''(\mathcal{D}_t w)} \\
&= -\overline{z_X'''(\widehat{\mathcal{D}}_t w + \mathbf{V}'' \cdot \nabla w)} \\
&= -\widehat{\mathcal{D}}_t(\overline{z_X''' w}) + \overline{(\widehat{\mathcal{D}}_t z_X''')w} - \overline{z_X''' \mathbf{V}'' \cdot \nabla w} \\
&= -\widehat{\mathcal{D}}_t(\overline{z_X''' w}) + \overline{w_X''' w} - \widehat{\mathbf{V}}_X \cdot \overline{(\nabla z''')w} - \widehat{\varpi}_X \overline{z_z''' w} - \overline{(\mathbf{V}'' \cdot \nabla z''')_X''' w} - \overline{z_X''' \mathbf{V}'' \cdot \nabla w} \\
&= -\widehat{\mathcal{D}}_t(\overline{z_X''' w}) - \overline{\pi}_X - \widehat{\mathbf{V}}_X \cdot \overline{(\nabla z''')w} - \widehat{\varpi}_X \overline{z_z''' w} + \overline{(\mathbf{V}'' \cdot \nabla z''')w_X'''} - \overline{z_X''' \mathbf{V}'' \cdot \nabla w} \\
&= -\widehat{\mathcal{D}}_t(\overline{z_X''' w'''}) - \overline{\pi}_X - \widehat{\mathbf{V}}_X \cdot \overline{(\nabla z''')w'''} - \widehat{\varpi}_X \overline{z_z''' w'''} - \overline{z_X''' \mathbf{V}''} \cdot \nabla \overline{w} + \overline{\mathbf{V}'' \cdot (w_X''' \nabla z''' - z_X''' \nabla w''')}, \qquad (20)
\end{aligned}
$$

where $\overline{\pi} \equiv -\overline{w''' w'''}/2 + \overline{(\mathbf{V}'' \cdot \nabla z''')w'''}$ is the Bernoulli head. Equation (20) has been developed from AM78 (Appendix A). On the other hand, with the vertical component of the momentum equation in mind, we substitute $A = w$ to (19) and $X = z$ to (20), and take the sum of the two equations to give

$$\begin{aligned} 0 &= \mathcal{R}\mathcal{S}^w - \overline{\mathbf{V}''' \cdot \nabla w'''} + \overline{z_z'''(\mathbf{V}''' - \mathbf{V}'')} \cdot \nabla \overline{w} - \overline{\pi}_z \\ &\quad - \widehat{\mathbf{V}}_z \cdot \overline{(\nabla z''')w'''} - \widehat{\varpi}_z \overline{z_z''' w'''} + \overline{\mathbf{V}'' \cdot (w_z''' \nabla z''' - z_z''' \nabla w''')}. \end{aligned} \tag{21}$$

The overall transformation in this section has been done without restricting the characteristics of the waves, and thus allows for finite-amplitude inhomogeneous unsteady waves.

3. Deriving the pseudomomentum in the VL framework

In this section, we show that the velocity-based expression for the form stress derived in the previous section contains the time derivative of a wave-induced velocity which might be called the pseudomomentum in the VL framework, as it is analogous to the generalized pseudomomentum in AM78. When the velocity-based expression of the form stress term is combined with the TWM momentum equations (6b,c), we obtain an expression for the time development of "the total transport velocity minus the VL pseudomomentum" which might be called the quasi-EM velocity in the VL framework.

Since our aim in this section is to show that our equation for the time development of the quasi-EM velocity in the VL framework includes the vortex force, we adopt the same scaling for the waves and the low-pass filtered flow (i.e. LCs) as in CL76. Let $\alpha \ll 1$ be a measure of the surface slope of waves. CL76 assumed that (i) $O(\alpha)$ waves are steady and monochromatic, (ii) the strength of the low-pass filtered flow is $O(\alpha^2)$, (iii) there is no separation between the wavelength of the waves and the horizontal scale of variation of the low-pass filtered flow, and that (iv) the time development of the low-pass filtered flow is two orders, in terms of α, slower than the phase cycle of the waves. The time derivative operator may then be decomposed as $\partial_t = \partial_\tau + \alpha^2 \partial_T$ where ∂_τ and ∂_T operates on wave and low-pass filtered quantities, respectively. These conditions are summarized in Table 3. Throughout the remainder of the manuscript we assume an infinitely deep ocean. [We have confirmed that the machinery of the present study is also applicable to a flat-bottomed ocean (not shown). See also footnote 6.]

a. Asymptotic expansion

To be consistent, the mean component of all quantities is scaled at $O(\alpha^2)$ except for $\overline{p}$ and $\overline{\eta}$ which are scaled at $O(\alpha^4)$,

$$\overline{z^{\varepsilon}} \equiv z, \tag{22a}$$

$$\overline{\eta} = \alpha^4 \overline{\eta}_4 + O(\alpha^5), \tag{22b}$$

$$\overline{p} = \alpha^4 \overline{p}_4 + O(\alpha^5), \tag{22c}$$

$$\overline{\mathbf{V}} = \alpha^2 \overline{\mathbf{V}}_2 + O(\alpha^3), \tag{22d}$$

$$\overline{w} = \alpha^2 \overline{w}_2 + O(\alpha^3), \tag{22e}$$

$$\widehat{\varpi} = \alpha^2 \widehat{\varpi}_2 + O(\alpha^3). \tag{22f}$$

The numeric subscripts represent the order of an asymptotic expansion, which is as in AG12 and AG13. Then we specialize to the $O(\alpha^2)$ terms in the TWM incompressibility equation (6a) as well as the $O(\alpha^4)$ terms in the TWM momentum equations (9a)-(9b) to yield,

$$\nabla \cdot \widehat{\mathbf{V}}_2 + \widehat{\varpi}_{2z} = 0, \tag{23a}$$

$$\widehat{\mathcal{D}}_T \widehat{\mathbf{V}}_2 + \mathcal{R}\mathcal{S}_4^{\mathbf{V}} = -\nabla(\overline{p}_4 + \overline{\eta}_4) + \mathcal{F}\mathcal{S}_4^{\mathbf{V}}, \tag{23b}$$

$$\widehat{\mathcal{D}}_T \widehat{w}_2 + \mathcal{R}\mathcal{S}_4^{w} = -\overline{p}_{4z}, \tag{23c}$$

where $\widehat{\mathcal{D}}_T \equiv \partial_T + \widehat{\mathbf{V}}_2 \cdot \nabla + \widehat{\varpi}_2 \partial_z$ is the material derivative operator based on the total transport velocity at $O(\alpha^2)$. On the other hand, the fluctuation component of all quantities is expanded from $O(\alpha)$, except for ϖ'' which is expanded from $O(\alpha^5)$,

$$z''' = \alpha z_1''' + \alpha^2 z_2''' + \alpha^3 z_3''' + O(\alpha^4), \tag{24a}$$

$$\eta''' = \alpha \eta_1''' + \alpha^2 \eta_2''' + \alpha^3 \eta_3''' + O(\alpha^4), \tag{24b}$$

$$p''' = \alpha p_1''' + \alpha^2 p_2''' + \alpha^3 p_3''' + O(\alpha^4), \tag{24c}$$

$$\mathbf{V}''' = \alpha \mathbf{V}_1''' + \alpha^2 \mathbf{V}_2''' + \alpha^3 \mathbf{V}_3''' + O(\alpha^4), \tag{24d}$$

$$w''' = \alpha w_1''' + \alpha^2 w_2''' + \alpha^3 w_3''' + O(\alpha^4), \tag{24e}$$

$$\varpi'' = O(\alpha^5). \tag{24f}$$

The scaling of ϖ'' stems from the scaling of mean sea surface height $\overline{\eta}$. This is because the kinematic boundary condition at the sea surface is $\varpi'' = \mathbf{V}'' \cdot \nabla \overline{\eta}$ (AG12) so that to the leading order, $\varpi_5'' = \mathbf{V}_1'' \cdot \nabla \overline{\eta}_4$ (or $\varpi_5''' = \mathbf{V}_1''' \cdot \nabla \overline{\eta}_4$ because $\mathbf{V}_1''' = \mathbf{V}_1''$).[6] Therefore the asymptotic expansion for ϖ'' starts from $O(\alpha^5)$, allowing us to formally ignore the last term of (7) to give

$$\mathcal{RS}_4^{\mathbf{V}} = \nabla \cdot (\overline{\mathbf{V}''\mathbf{V}''})_4 + \nabla \cdot (\overline{z_z'''\mathbf{V}''\mathbf{V}''})_4, \tag{25a}$$

$$\mathcal{RS}_4^{w} = \nabla \cdot (\overline{\mathbf{V}''w''})_4 + \nabla \cdot (\overline{z_z'''\mathbf{V}''w''})_4. \tag{25b}$$

The numeric subscript attached to the brackets represents summation of terms at a given order of α (see Table 4 for a template). To obtain the $O(\alpha^4)$ form stress term for use in (23b), we use the velocity-based

[6] In contrast to the present study, the quantity $\varpi''(= w^{*\prime\prime})$ has been scaled at $O(\alpha^2)$ in AG13 because they considered waves in shallow water on a bottom slope of $O(\alpha)$. See Eqs. (6a-c) of AG13 for the explicit expressions of the kinematic boundary conditions at a sloping bottom in the VL framework (which should be compared with bottom boundary conditions in the three-dimensional Lagrangian framework in a future study). As shown by AG13, it is certainly possible to manipulate $\varpi''(= w^{*\prime\prime})$ even if it has been scaled at $O(\alpha^2)$. However, for simplicity, the present study assumes an infinitely deep ocean (or a flat-bottomed ocean) and thus neglects ϖ'' from (19) and thereafter.

expression (12) to give

$$\mathcal{FS}_4^{\mathbf{V}} = [\overline{z_z'''(\mathcal{D}_t\mathbf{V})'''} - \overline{\nabla z'''(\mathcal{D}_t w)'''}]_4. \tag{26}$$

To summarize both the Reynolds stress term and the form stress term consist of the effect of waves up to $O(\alpha^3)$, as can be seen from Table 4.

Substitution of (22a)-(22f) and (24a)-(24f) to (4a)-(4d) yields

$$z_{1z\tau}''' + \nabla \cdot \mathbf{V}_1''' = 0, \tag{27a}$$

$$z_{1\tau}''' = w_1''', \tag{27b}$$

$$\mathbf{V}_{1\tau}''' = -\nabla(p_1''' + \eta_1'''), \tag{27c}$$

$$w_{1\tau}''' = -p_{1z}''', \tag{27d}$$

at $O(\alpha)$ and

$$z_{2z\tau}''' + \nabla \cdot (\mathbf{V}_2''' + z_{1z}'''\mathbf{V}_1''') = 0, \tag{28a}$$

$$z_{2\tau}''' + \mathbf{V}_1''' \cdot \nabla z_1''' = w_2''', \tag{28b}$$

$$\mathbf{V}_{2\tau}''' + \mathbf{V}_1''' \cdot \nabla \mathbf{V}_1''' = -\nabla(p_2''' + \eta_2''') - w_{1\tau}'''\nabla z_1''', \tag{28c}$$

$$w_{2\tau}''' + \mathbf{V}_1''' \cdot \nabla w_1''' = -p_{2z}''' - w_{1\tau}''' z_{1z}''', \tag{28d}$$

at $O(\alpha^2)$. Note that since the first order waves are steady and horizontally homogeneous, it follows from (28a)-(28d) that the second order waves are also steady and horizontally homogeneous. The effect of the low-pass filtered flow, e.g. LCs, on the waves appears in equations for $O(\alpha^3)$ waves, to be explained later in the manuscript.

b. First and second order waves

As assumed above, $O(\alpha)$ waves are monochromatic and steady: $\eta_1''' = \mathcal{A}\cos\theta$ where $\mathcal{A}$ is wave amplitude, $\theta = kx + ly - \sigma\tau$ is wave phase with k and l being wavenumbers in the $x-$ and $y-$direction, and σ is wave frequency. These parameters are constant on the time and spatial scales of waves (i.e. $\partial_\tau A = 0$ and $\nabla A = 0$ for $A = \mathcal{A}, k, l, \sigma$) which leads to

$$\partial_\tau(\overline{A_1'''B_1'''}) = 0, \qquad \overline{A_1'''B_{1\tau}'''} = -\overline{A_{1\tau}'''B_1'''}, \tag{29a}$$

$$\nabla(\overline{A_1'''B_1'''}) = 0, \qquad \overline{A_1'''\nabla B_1'''} = -\overline{(\nabla A_1''')B_1'''}, \tag{29b}$$

where A_1''' and B_1''' are wave variables at $O(\alpha)$. The sea surface is located at $z^\varepsilon = \overline{\eta} + \eta'''$ in the Eulerian-Cartesian coordinates, and is located at $z = \overline{\eta} = \alpha^4\overline{\eta}_4 + O(\alpha^5)$ in the VL coordinates. As far as up to $O(\alpha^3)$ waves are concerned, $z = 0$ can be used as the label of sea surface in the VL coordinates. With the boundary conditions of $\mathbf{V}_1''' = 0$ at $z = -\infty$ and $p_1''' = 0$ at $z = 0$, we solve (27a)-(27d) to yield,

$$\sigma^2 = \kappa, \quad \kappa \equiv \sqrt{k^2 + l^2}, \tag{30a}$$

$$\phi_1''' \equiv (\mathcal{A}/\kappa)(\exp\kappa z)\cos\theta, \tag{30b}$$

$$\mathbf{V}_1''' = \nabla\phi_{1\tau}''' = \sigma(\nabla\theta)\phi_1''', \tag{30c}$$

$$w_1''' = \phi_{1z\tau}''' = -\sigma\phi_{1z\theta}''', \tag{30d}$$

$$z_1''' = \phi_{1z}''', \tag{30e}$$

$$p_1''' = \sigma^2\phi_1''' - \eta_1''', \tag{30f}$$

where $\nabla\phi_1''' = (\nabla\theta)\phi_{1\theta}'''$, $\phi_{1\tau}''' = -\sigma\phi_{1\theta}'''$, $\nabla\theta = (k, l)$ and $\eta_1''' = z_1'''|_{z=0}$ are understood. The above solution is given in the VL coordinates. Then we compute the quasi-Stokes velocity using (10a)-(10b). For an arbitrary quantity A, the TWM and the EM at $O(\alpha^2)$ can be written as

$$\widehat{A}_2 = \overline{A}_2 + \overline{z'''_{1z} A'''_1}, \tag{31a}$$

$$\overline{A}^{\varepsilon}_2 = \overline{A}_2 - \overline{z'''_1 A'''_{1z}}, \tag{31b}$$

where (31b) is given by a Taylor expansion in the vertical direction.[4] Substitution of (30c)-(30e) and (31a)-(31b) to (10a)-(10b) yields,

$$\mathbf{V}^{qs}_2 = \overline{z'''_{1z}\mathbf{V}'''_1} + \overline{z'''_1\mathbf{V}'''_{1z}} = (\overline{z'''_1\mathbf{V}'''_1})_z = (\overline{\phi'''_{1z}\phi'''_1})_z \sigma\nabla\theta, \tag{32a}$$

$$w^{qs}_2 = -\overline{\mathbf{V}'''_1 \cdot \nabla z'''_1} + \overline{z'''_1 w'''_{1z}} = -\nabla\cdot(\overline{z'''_1\mathbf{V}'''_1}) = 0, \tag{32b}$$

which has been shown by Mellor (2003) and Smith (2006), except that these authors did not refer to the term "quasi-Stokes velocity". Equation (32b) indicates that the condition of horizontally homogeneous waves leads to $w^{qs}_2 = 0$, namely $\widehat{\varpi}_2 = \overline{w}^{\varepsilon}_2$.

Substitution of the $O(\alpha)$ solution (30c)-(30e) to $O(\alpha^2)$ momentum equations (28c)-(28d) yields,

$$\mathbf{V}'''_{2\tau} = -\nabla(p'''_2 + \eta'''_2), \tag{33a}$$

$$w'''_{2\tau} = -\partial_z(p'''_2 + \eta'''_2), \tag{33b}$$

which indicates that $O(\alpha^2)$ velocity (as well as $O(\alpha)$ velocity) satisfies an apparent[7] irrotational condition in the VL coordinates:

$$\nabla\times\mathbf{V}'''_i = 0, \quad \nabla w'''_i - \partial_z\mathbf{V}'''_i = 0, \tag{34}$$

for $i = 1$ and 2, where here, $\nabla\times\mathbf{V}''' = (v'''_x - u'''_y)\mathbf{z}$ and $\mathbf{z}$ is a unit vector in the upwards vertical direction.

[7]Although (34) is sufficient for us to derive the vortex force, the standard vorticity defined in the three-dimensional Eulerian coordinates reads $\nabla^{\varepsilon}\times\mathbf{V} = (z^{\varepsilon}_z\nabla\times\mathbf{V} - \nabla z^{\varepsilon}\times\mathbf{V}_z)/z^{\varepsilon}_z$ and $\nabla^{\varepsilon}w - \mathbf{V}_{z^{\varepsilon}} = (z^{\varepsilon}_z\nabla w - w_z\nabla z^{\varepsilon} - \mathbf{V}_z)/z^{\varepsilon}_z$.

c. Substitution of the velocity-based form stress term

The condition of horizontally homogeneous and steady waves is limited to up to $O(\alpha^2)$. The Doppler effect by both the horizontal and vertical circulations associated with the low-pass filtered flow appears when considering the waves at $O(\alpha^3)$. However it is rather difficult to derive (i) an analytical solution for $O(\alpha^3)$ waves and (ii) a depth-dependent wave crest equation. It is the depth-independent wave crest equation that has been used in Garrett (1976) and Smith (2006). Leibovich (1980) avoids these difficulties using the Lagrangian mean framework, an approach we mimic here. In fact the two terms on the last line of (12) have been expanded using (4a) and (4b), respectively, to give (19) and (20) (see Section 2c). We pick up the $O(\alpha^4)$ terms of (19)-(20) and delete some terms, using the phase relationship of $O(\alpha)$ waves in (30c)-(30e), to give

$$\mathcal{FS}_4^u = \widehat{\mathcal{D}}_T(\overline{z_{1z}'''u_1'''} - \overline{z_{1x}'''w_1'''}) + \overline{z_{1z}'''\mathbf{V}_1'''} \cdot \nabla\overline{u}_2 - \widehat{\mathbf{V}}_{2x} \cdot \overline{(\nabla z_1''')w_1'''} + \mathcal{RS}_4^u - \overline{\pi}_{4x} - (\overline{\mathbf{V}''' \cdot \nabla u'''})_4, \quad (35a)$$

$$\mathcal{FS}_4^v = \widehat{\mathcal{D}}_T(\overline{z_{1z}'''v_1'''} - \overline{z_{1y}'''w_1'''}) + \overline{z_{1z}'''\mathbf{V}_1'''} \cdot \nabla\overline{v}_2 - \widehat{\mathbf{V}}_{2y} \cdot \overline{(\nabla z_1''')w_1'''} + \mathcal{RS}_4^v - \overline{\pi}_{4y} - (\overline{\mathbf{V}''' \cdot \nabla v'''})_4, \quad (35b)$$

where $\overline{\pi}_4 \equiv -(\overline{w'''^2})_4/2 + [\overline{(\mathbf{V}'' \cdot \nabla z''')w'''}]_4$ is the Bernoulli head (to be updated later in the manuscript). [8] Substitution of (35a)-(35b) to (23b) yields

$$\widehat{\mathcal{D}}_T(\overline{u}_2 + \overline{z_{1x}'''w_1'''}) = -(\overline{p}_4 + \overline{\eta}_4 + \overline{\pi}_4)_x + \overline{z_{1z}'''\mathbf{V}_1'''} \cdot \nabla\overline{u}_2 - \widehat{\mathbf{V}}_{2x} \cdot \overline{(\nabla z_1''')w_1'''} - (\overline{\mathbf{V}''' \cdot \nabla u'''})_4, \quad (36a)$$

$$\widehat{\mathcal{D}}_T(\overline{v}_2 + \overline{z_{1y}'''w_1'''}) = -(\overline{p}_4 + \overline{\eta}_4 + \overline{\pi}_4)_y + \overline{z_{1z}'''\mathbf{V}_1'''} \cdot \nabla\overline{v}_2 - \widehat{\mathbf{V}}_{2y} \cdot \overline{(\nabla z_1''')w_1'''} - (\overline{\mathbf{V}''' \cdot \nabla v'''})_4, \quad (36b)$$

where the effect of $O(\alpha^3)$ waves appear in both the Bernoulli head $\overline{\pi}_4$ and the last term of each equation. The last term of (36a)-(36b) is the legacy of the Reynolds stress term, and is absent in the three-dimensional Lagrangian framework of AM78. It should be noted that the $\widehat{\mathcal{D}}_T\widehat{\mathbf{V}}_2$ term on the lhs of

[8] The last term on the last line of (20) becomes $\overline{v''(\nabla w''' \times \nabla z''')}$ in the case of $X = x$, and $-\overline{u''(\nabla w''' \times \nabla z''')}$ in the case of $X = y$. When taking an asymptotic expansion of these terms, $\nabla w_i''' \times \nabla z_j''' = 0$ for $(i, j) = (1, 1)$, $(1, 2)$, and $(2, 1)$. This is because $\nabla w_i''' \propto \nabla\theta$ and $\nabla z_i''' \propto \nabla\theta$ according to the analytical solution of $O(\alpha)$ and $O(\alpha^2)$ waves.

(23b) has been partially cancelled by the $\widehat{\mathcal{D}}_T$ terms on the rhs of (35a)-(35b) using $\widehat{\mathbf{V}}_2 = \overline{\mathbf{V}}_2 + \overline{z'''_{1z}\mathbf{V}'''_1}$ which follows from (31a). This is the first indication of the appearance of the pseudomomentum, to be discussed in detail in the next subsection.

A nice feature of AM78 is that the Bernoulli head is present in both the horizontal and vertical components of the wave-averaged momentum equations, and there is no need to treat the Bernoulli head and nonhydrostatic pressure separately (cf. Craik, 1985; Dingemans, 2009). The sum of the Bernoulli head and nonhydrostatic pressure can be obtained by solving a Poisson equation based on the incompressibility condition of circulation (as is always the case in nonhydrostatic numerical models). Eventually there is no need to derive the analytical solution for $O(\alpha^3)$ waves. It is therefore very useful to mimic this feature of AM78 – the Bernoulli head appearing next to nonhydrostatic pressure in both the horizontal and vertical components of the wave-averaged momentum equations. So far the Bernoulli head in our analysis appears only in the horizontal component of the TWM momentum equations (36a)-(36b). However, it is straightforward to write the vertical momentum equation in a form that includes the Bernoulli head. Indeed, we have derived (21) which allows us to transform the Reynolds stress term in the vertical component of the momentum equation (23c), into the sum of the vertical gradient of the Bernoulli head and the other terms, as follows,

$$\widehat{\mathcal{D}}_T(\underbrace{\overline{w}_2 + \overline{z'''_{1z}w'''_1}}_{\widehat{w}_2}) = -(\overline{p}_4 + \overline{\pi}_4)_z - \widehat{\mathbf{V}}_{2z} \cdot \overline{(\nabla z'''_1)w'''_1} + [\overline{\mathbf{V}'' \cdot (w'''_z \nabla z''' - z'''_z \nabla w''')'''}]_4 - (\overline{\mathbf{V}''' \cdot \nabla w'''})_4, \quad (36c)$$

where the expression for $\widehat{w}_2$ uses (31a). Note that $O(\alpha^3)$ waves appear in both the Bernoulli head $\overline{\pi}_4$ and the last term. The second last term of (36c) consists of waves up to $O(\alpha^2)$ that are horizontally homogeneous (so that, as we shall show later, the term can be absorbed into the Bernoulli head without affecting the horizontal component of the momentum equations).

d. The pseudomomentum in the VL framework

Interestingly the three-dimensional components of velocity on the lhs of (36a)-(36c) have a symmetry, and can be written as the difference of the total transport velocity and a wave-induced velocity,

$$\begin{pmatrix} \overline{u}_2 + \overline{z'''_{1x} w'''_1} \\ \overline{v}_2 + \overline{z'''_{1y} w'''_1} \\ \overline{w}_2 + \overline{z'''_{1z} w'''_1} \end{pmatrix} = \underbrace{\begin{pmatrix} \widehat{u}_2 \\ \widehat{v}_2 \\ \widehat{\varpi}_2 \end{pmatrix}}_{\text{total transport velocity}} - \underbrace{\overline{\begin{pmatrix} +z'''_{1z} & 0 & -z'''_{1x} \\ 0 & +z'''_{1z} & -z'''_{1y} \\ -z'''_{1x} & -z'''_{1y} & -z'''_{1z} \end{pmatrix} \begin{pmatrix} u'''_1 \\ v'''_1 \\ w'''_1 \end{pmatrix}}}_{\text{pseudomomentum}}, \tag{37}$$

where (31a) and (17) have been used. The wave-induced velocity on the rhs may be called the pseudomomentum in the VL framework, as it is analogous to the generalized pseudomomentum in AM78 (Appendix A). By substituting (30c)-(30e) to the last term of (37), we confirm that the content of the pseudomomentum is identical to the quasi-Stokes velocity in (32a)-(32b). This association of the pseudomomentum in (37) to the quasi-Stokes velocity in (32a)-(32b) stems from the conditions of (i) "apparent" irrotational wave motions as given by (34) and (ii) horizontally homogeneous and steady waves as given by (29a)-(29b). In fact

$$\begin{aligned} \mathbf{V}_2^{qs} &= \overline{z'''_{1z}\mathbf{V}'''_1} + \overline{z'''_1\mathbf{V}'''_{1z}} \\ &= \overline{z'''_{1z}\mathbf{V}'''_1} + \overline{z'''_1\nabla w'''_1} \\ &= \overline{z'''_{1z}\mathbf{V}'''_1} - \overline{(\nabla z'''_1)w'''_1}, \end{aligned} \tag{38a}$$

$$\begin{aligned} w_2^{qs} &= -\overline{\mathbf{V}'''_1 \cdot \nabla z'''_1} + \overline{z'''_1 w'''_{1z}} \\ &= -\overline{\mathbf{V}'''_1 \cdot \nabla z'''_1} + \overline{z'''_1 z'''_{1z\tau}} \\ &= -\overline{\mathbf{V}'''_1 \cdot \nabla z'''_1} - \overline{w'''_1 z'''_{1z}}, \end{aligned} \tag{38b}$$

where (27b) has been used in (38b). The last line of each of (38a)-(38b) is identical to the definition of the pseudomomentum in (37). Recall that the difference of the total transport velocity and the quasi-Stokes velocity is the EM velocity. Hence the lhs of (36a)-(36c) is the same as $\widehat{\mathcal{D}}_T\overline{u}_2^{\varepsilon}$, $\widehat{\mathcal{D}}_T\overline{v}_2^{\varepsilon}$, and $\widehat{\mathcal{D}}_T\overline{w}_2^{\varepsilon}$: the material derivative of the EM velocity.

To summarise, in this section we have shown how to extract the pseudomomentum in the VL framework from the expression for the form stress written in terms of velocity rather than pressure. It is now straightforward, although mathematically laborious, to derive the expression for the vortex force in the VL framework. This is done in Appendix B.

4. The effect of viscosity

So far the analysis in the present study has been done for waves in a nonrotating inviscid fluid. As shown below, the velocity-based expression of the form stress term (12) is modified by the introduction of the Coriolis term and the viscosity term.

a. The form stress term for a rotating viscid fluid

Inclusion of the new terms to the momentum equations (11a)-(11b) in the VL coordinates yields,

$$\mathcal{D}_t\mathbf{V} + f\mathbf{z}\times\mathbf{V} = -\nabla(p+\eta) - \underbrace{(\mathcal{D}_t w - F^w)}_{-p_{z^\varepsilon}}\nabla z''' + F^{\mathbf{V}}, \tag{39a}$$

$$\underbrace{(1+z_z''')}_{z_z^\varepsilon}\mathcal{D}_t w = -p_z + \underbrace{(1+z_z''')}_{z_z^\varepsilon}F^w, \tag{39b}$$

where $(1+z_z''')p_{z^\varepsilon} = z_z^\varepsilon p_{z^\varepsilon} = p_z$ is understood. The symbols $F^{\mathbf{V}}$ and F^w represent the effect of turbulent mixing on $\mathbf{V}$ and w, respectively. These terms are parameterized using a conventional symmetric tensor in Eulerian-Cartesian coordinates, as in Eq. (28) of AG12. The Coriolis parameter f as well as the coefficient of turbulent viscosity ν (to appear later) have been nondimensionalized following the approach of AG13.[9]

[9]The nondimensionalization is written by $\acute{f} = (\acute{g}/\acute{\Delta})^{1/2}f$ and $\acute{\nu} = (\acute{\Delta}^3\acute{g})^{1/2}\nu$ using the notation in Appendix A of AG13.

The TWM equation system (6a)-(6c) becomes

$$
\begin{aligned}
\nabla \cdot \widehat{\mathbf{V}} + \widehat{\varpi}_z &= 0, && (40a)\\
\widehat{\mathbf{V}}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{\mathbf{V}}) + (\widehat{\varpi}\widehat{\mathbf{V}})_z + f\mathbf{z} \times \widehat{\mathbf{V}} + \mathcal{RS}^{\mathbf{V}} &= -\nabla(\overline{p} + \overline{\eta}) + \mathcal{FS}^{\mathbf{V}} + \widehat{F}^{\mathbf{V}}, && (40b)\\
\widehat{w}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{w}) + (\widehat{\varpi}\widehat{w})_z + \mathcal{RS}^{w} &= -\overline{p}_z + \widehat{F}^{w}, && (40c)
\end{aligned}
$$

where the Coriolis term includes $f\mathbf{z} \times \mathbf{V}^{qs}$ which corresponds to the Coriolis-Stokes force of Hasselmann (1970) and Huang (1979). The Reynolds and form stress terms in (40b)-(40c) are the same as (7) and (8), respectively. The velocity-based expression of the form stress term (12) is revised as follows,

$$
\begin{aligned}
\mathcal{FS}^{\mathbf{V}} &= -\overline{z'''_z \nabla(p + \eta)} + \overline{(\nabla z''')p_z}\\
&= \overline{z'''_z[\mathcal{D}_t\mathbf{V} + f\mathbf{z} \times \mathbf{V} + (\mathcal{D}_t w - F^w)\nabla z''' - F^{\mathbf{V}}]} - \overline{\nabla z'''(1 + z'''_z)(\mathcal{D}_t w - F^w)}\\
&= \overline{z'''_z(\mathcal{D}_t\mathbf{V})'''} - \overline{\nabla z'''(\mathcal{D}_t w)'''} + f\mathbf{z} \times \overline{z'''_z\mathbf{V}'''} - \overline{z'''_z(F^{\mathbf{V}})'''} + \overline{(\nabla z''')(F^w)'''}, \qquad (41)
\end{aligned}
$$

where (39a)-(39b) have been used.

b. Asymptotic expansion

The derivation leading to (41) was obtained without approximation. We now specialize to small amplitude waves.

Jenkins (1987) investigated the problem of how the presence of surface waves modifies the classical Ekman spiral solution, and obtained an interesting result when the viscosity coefficient varies in the vertical direction (to be explained later). We wish to make the link to the work of Jenkins (1987), and thus retain the last three terms of (41) as follows. Some scalings typical for the ocean are $f/\sigma \sim O(\alpha^4)$ and $\nu\kappa^2/\sigma \sim O(\alpha^4)$ (see Table 2 of AG12 for the dimensional values of f, σ, κ, and ν). Thus the Coriolis parameter and the viscosity coefficient may be scaled as $f = \alpha^4 f_4$ and $\nu = \alpha^4\nu_4$, which indicates that the form stress term (41) should be written at $O(\alpha^6)$. For simplicity, we consider circulation whose variation is scaled as $m = n$ in Appendix C, and is associated with the equation system (C4a)-(C4c). Noting that the form stress term in (C4b) is written at $O(\alpha^{n+2})$, we obtain $m = n = 4$, namely $\partial_t = \partial_\tau + \alpha^4 \partial_T$ and $\nabla = \dot{\nabla} + \alpha^4 \overline{\nabla}$ where $\dot{\nabla}$ and $\overline{\nabla}$ are the lateral gradient operator for wave and low-pass filtered quantities, respectively. See Appendix C for details. These conditions are summarized in Table 3.

In what follows we consider depths below the base of the thin viscous surface boundary layer (of a few centimeters depth) associated with waves (hereafter TVSBL). Thus the $O(\alpha)$ quantities are the solution of inviscid waves (30a)-(30f). As in Section 3, $O(\alpha)$ waves are assumed to be monochromatic, except that slow variations in both the horizontal direction and in time are allowed as in (C1a)-(C1b) (see footnote C1). We first note that the viscosity term of the TWM momentum equation (40b) is written by $\widehat{F}_6^{\mathbf{V}} = \partial_z(\nu_4 \overline{\mathbf{V}}_{2z}^{\varepsilon})$, indicating that the viscosity acts on the EM velocity rather than the TWM velocity. This has been shown by Eqs. (30) and (44) of AG12 and note that the result holds even if the viscosity coefficient varies in the vertical.

Coming back to the velocity-based form stress term, we substitute (19)-(20) to (41), and then pick-up $O(\alpha^6)$ terms to yield,

$$\begin{aligned} \mathcal{FS}_6^{\mathbf{V}} &= \partial_T \mathbf{V}_2^{qs} + \mathcal{RS}_6^{\mathbf{V}} - \frac{1}{2}\overline{\nabla}(\overline{|\mathbf{V}_1'''|^2 - w_1'''^2}) - \overline{[(\nabla \times \mathbf{V}''') \times \mathbf{V}''']_6} + f_4 \mathbf{z} \times \overline{z_{1z}''' \mathbf{V}_1'''} \\ &\quad - \overline{z_{1z}'''(F^{\mathbf{V}})_5'''} + \overline{(\dot{\nabla} z_1''')(F^w)_5'''}, \end{aligned} \tag{42}$$

where (38a) and (B5a) have been used. The symbols $(F^{\mathbf{V}})_5'''$ and $(F^w)_5'''$ are the viscosity terms that are written using the solution of $O(\alpha)$ waves,

$$
\begin{aligned}
(F^u)_5''' &= [2\nu_4 u_{1x}''']_x + [\nu_4(v_{1x}''' + u_{1y}''')]_y + [\nu_4(w_{1x}''' + u_{1z}''')]_z \\
&= k\sigma[2\nu_4\dot{\nabla}^2\phi_1''' + (2\nu_4\phi_{1z}''')_z], \qquad (43a) \\
(F^v)_5''' &= [\nu_4(u_{1y}''' + v_{1x}''')]_x + [2\nu_4 v_{1y}''']_y + [\nu_4(w_{1y}''' + v_{1z}''')]_z \\
&= l\sigma[2\nu_4\dot{\nabla}^2\phi_1''' + (2\nu_4\phi_{1z}''')_z], \qquad (43b) \\
(F^w)_5''' &= [\nu_4(u_{1z}''' + w_{1x}''')]_x + [\nu_4(v_{1z}''' + w_{1y}''')]_y + [\nu_4 2w_{1z}''']_z \\
&= [2\nu_4\dot{\nabla}^2 w_1''' + (2\nu_4 w_{1z}''')_z], \qquad (43c)
\end{aligned}
$$

where the second line of each has been derived using (30b)-(30d) and $\partial_x\nu_4 = \partial_y\nu_4 = 0$. The viscosity coefficient ν_4 is allowed to vary in the vertical direction, which is similar to Jenkins (1987).

We now consider how to calculate the fourth term on the rhs of (42) which consists of waves up to $O(\alpha^5)$. The solution of higher order waves may be decomposed into that associated with the nonlinear terms of (39a)-(39b) and that associated with the effect of the Coriolis and viscosity terms. The former solution is written in terms of the harmonics of $O(\alpha)$ waves (not shown) and averages to zero and thus is not discussed further (this approach follows Section 4b of AG13). Hereafter we focus on the latter solution which is derived from the linear terms of (39a) to read,

$$
\partial_\tau \mathbf{V}_5''' + \partial_T \mathbf{V}_1''' + f_4\mathbf{z}\times\mathbf{V}_1''' = -\dot{\nabla}p_5''' - \overline{\nabla}p_1''' + (F^{\mathbf{V}})_5'''. \qquad (44)
$$

We take the curl of (44) to yield,

$$
\begin{aligned}
\dot{\nabla}\times(\partial_\tau\mathbf{V}_5''') + \dot{\nabla}\times(f_4\mathbf{z}\times\mathbf{V}_1''') &= -\dot{\nabla}\times\overline{\nabla}(p_1''' + \eta_1''') \\
&= \overline{\nabla}\times\dot{\nabla}(p_1''' + \eta_1''') \\
&= -\overline{\nabla}\times(\partial_\tau\mathbf{V}_1'''), \qquad (45)
\end{aligned}
$$

where the last term of (44) has canceled out because $(F^{\mathbf{V}})_5''' \propto \dot{\nabla}\theta = (k, l)$ which follows from (43a)-(43b). The last line of (45) has been derived using (27c). Using (45), $\dot{\nabla}\cdot\mathbf{V}_1''' + w_{1z}''' = 0$ and $w_1''' = z_{1\tau}'''$, we show

$$
\begin{aligned}
[(\nabla\times\mathbf{V}''')\times\mathbf{V}''']_6 &= (\dot\nabla\times\mathbf{V}_5''')\times\mathbf{V}_1''' + (\overline{\nabla}\times\mathbf{V}_1''')\times\mathbf{V}_1''' \\
&= \int^{\tau}[\dot\nabla\times(\partial_\tau\mathbf{V}_5''') + \overline{\nabla}\times(\partial_\tau\mathbf{V}_1''')]d\tau\times\mathbf{V}_1''' \\
&= -\int^{\tau}\dot\nabla\times(f_4\mathbf{z}\times\mathbf{V}_1''')d\tau\times\mathbf{V}_1''' \\
&= \int^{\tau} f_4\mathbf{z}w_{1z}'''d\tau\times\mathbf{V}_1''' \\
&= f_4\mathbf{z}\times z_{1z}'''\mathbf{V}_1''', \qquad (46)
\end{aligned}
$$

where the first line has been derived using $\dot\nabla\times\mathbf{V}_i'''=0$ for $i=1,2,3,4$. Equation (46) indicates that the fourth and fifth terms on the rhs of (42) cancel each other.

We then calculate the viscosity terms of (42). The last two terms of (42) may be rewritten using $z_{1z}'''=\phi_{1zz}'''=\kappa^2\phi_1'''=(\dot\nabla\theta\cdot\dot\nabla\theta)\phi_1'''=(1/\sigma)(\mathbf{V}_1'''\cdot\dot\nabla\theta)$ and $\dot\nabla z_1'''=(\dot\nabla\theta)z_{1\theta}'''=(-1/\sigma)(\dot\nabla\theta)w_1'''$ to yield

$$
\begin{aligned}
-\overline{z_{1z}'''(F^{\mathbf{V}})_5'''}+\overline{(\dot\nabla z_1''')(F^w)_5'''} &= -\frac{1}{\sigma}\overline{(\mathbf{V}_1'''\cdot\dot\nabla\theta)(F^{\mathbf{V}})_5'''}-\frac{\dot\nabla\theta}{\sigma}\overline{w_1'''(F^w)_5'''} \\
&= -\frac{\dot\nabla\theta}{\sigma}\underbrace{\left[\overline{\mathbf{V}_1'''\cdot(F^{\mathbf{V}})_5'''}+\overline{w_1'''(F^w)_5'''}\right]}_{\text{FluxDiv}-\text{Dissipation}}, \qquad (47)
\end{aligned}
$$

where the last line has been derived using $(F^{\mathbf{V}})_5'''=(2\sigma\dot\nabla\theta)[\nu_4\dot\nabla^2\phi_1'''+(\nu_4\phi_{1z}''')_z]$ which follows from (43a)-(43b). Substitution of (46)-(47) to (42) yields,

$$
\mathcal{FS}_6^{\mathbf{V}} = \partial_T\mathbf{V}_2^{qs}+\mathcal{RS}_6^{\mathbf{V}}-\frac{1}{2}\overline{\nabla}(\overline{|\mathbf{V}_1'''|^2-w_1'''^2})-\frac{\dot\nabla\theta}{\sigma}\underbrace{\left[\overline{\mathbf{V}_1'''\cdot(F^{\mathbf{V}})_5'''}+\overline{w_1'''(F^w)_5'''}\right]}_{\text{FluxDiv}-\text{Dissipation}}, \qquad (48)
$$

where the last term is the product between $-(\dot\nabla\theta)/\sigma$ and the viscosity term of the depth-dependent energy equation, the latter of which may be separated into two terms: one is the vertical divergence of a viscosity-induced flux (noted as FluxDiv) and one is dissipation at depths below $z=\overline{\eta}-\delta$ (noted as Dissipation) where $\delta(>0)$ is the thickness of the TVSBL. This separation, in particular the identification of the dissipation term in the wave energy equation, is based on Phillips (1977, page 52). See Appendix D of the present manuscript for details. Substitution of (48) to either (40b) or the rotating viscid version of (C4b) yields,

$$\partial_T \overline{\mathbf{V}}_2^{\varepsilon} + f_4 \mathbf{z}\times\widehat{\mathbf{V}}_2 = -\overline{\nabla}[\overline{p}_2 + \overline{\eta}_2 + (\overline{|\mathbf{V}_1'''|^2 - w_1'''^2})/2] + \underbrace{\partial_z(\nu_4 \overline{\mathbf{V}}_{2z}^{\varepsilon})}_{\widehat{F}_6^{\mathbf{V}}} + \underbrace{\partial_z(-\nu_4\mathbf{V}_{2z}^{qs})}_{(-\dot{\nabla}\theta/\sigma)\mathsf{FluxDiv}} + \underbrace{\nu_4 \mathbf{V}_{2zz}^{qs}}_{(\dot{\nabla}\theta/\sigma)\mathsf{Dissipation}}, \qquad (49)$$

where the last two terms have been derived using (32a) and (D1a)-(D1b).

The last term in (49) originates from the dissipation term in the wave energy equation (D2) and acts like a depth-dependent body force, indicating that the dissipation of wave kinetic energy leads to transfer of momentum from waves to circulation (Fig. 2). The second to last term of (49) is associated the vertical flux of kinetic energy in the depth-dependent wave energy equation. It is important to remember from the outset that the last two terms on the rhs of (49) originate from the form stress term. As such they represent the effect of the waves on circulation, and have a vertical structure. In contrast to the present study, the explanation in Smith (2006) and Weber et al. (2006) is based on depth-integrated equations.

Once the last two terms of (49) are merged, the equation is identical to Eq. (5.1) of Jenkins (1987) who obtained it from the three-dimensional Lagrangian framework of Pierson (1962). Indeed, when the viscosity varies in the vertical, merging the last two terms of (49) leads to the “additional source of momentum” in the water column noted by Jenkins (1987). Nevertheless, the surface boundary condition derived by Jenkins (1987) is different from that appropriate to our study (see Section 4c below). As explained in footnote B1 of AG12, we believe the boundary condition used by Jenkins (1987) (and also by Weber (1983)) is not correct.

c. The virtual wave stress of Longuet-Higgins (1953, 1960)

We now consider the surface boundary condition of the momentum equation (49), assuming that the net momentum flux through the TVSBL is vertically uniform. In doing so, we implicitly make use of the fact that the TVSBL, being only centimeters thick, is much thinner than the Ekman layer associated with the rotation of the Earth, typically measurable in meters, or even 10's of metres - see Table 2 in AG12.

Combining the vertical flux of momentum associated with the second and third last terms of (49) yields $\nu_4(\overline{\mathbf{V}}_2^\varepsilon - \mathbf{V}_2^{qs})_z$ at the base of the base of the TVSBL ($z = \overline{\eta} - \delta$). This momentum flux should match the skin viscous stress applied at the top of the TVSBL, namely the ocean surface ($z = \overline{\eta}$).[10] If there is no wind, the skin stress at the ocean surface is zero. Therefore the viscosity-induced momentum flux at the base of the TVSBL is also zero: $\nu_4(\overline{\mathbf{V}}_2^\varepsilon - \mathbf{V}_2^{qs})_z = 0$ at $z = \overline{\eta} - \delta$. The result that $\overline{\mathbf{V}}_{2z}^\varepsilon = \mathbf{V}_{2z}^{qs}$ is consistent with Longuet-Higgins (1953, 1960) who found that the vertical gradient of the LM velocity at the base of the TVSBL is twice that of the Stokes-drift velocity. It follows that the viscosity-induced stress $\nu_4 \mathbf{V}_{2z}^{qs}$ corresponds to the VWS. The explanation in the present study makes it clear that this result of Longuet-Higgins is more general than it might appear. In particular, the role played by the VWS has emerged without the explicit use of the analytical solution of waves in the TVSBL, and thus is easily applicable to various types of problem. It does not matter whether there is wind forcing or not, the waves can have slow variations, and the viscosity coefficient can vary in the vertical - the result is quite general. Indeed, (49) represents an extension beyond the approach of Ünlüata and Mei (1970), Weber (1983), Xu and Bowen (1994), Ng (2004), and AG12 based on the analytical solution of waves including the TVSBL. The above explanation of the VWS also works for the scaling for the vortex force equations in Section 3 (not shown).

Furthermore, (49) provides a prescription for including surface wave effects in ocean circulation

[10]The skin stress at the sea surface corresponds to the mean tangential stress $\overline{\tau}$ in AG12 (noting that τ in the present study has a different meaning and represents the time measure associated with the phase cycle of the waves). The definitions of the skin stress and the wave stress are partly related and require care, a topic we shall discuss in a later paper.

models. At the sea surface ($z = \overline{\eta}$), the net momentum flux from air (i.e. wind) to water (i.e. ocean circulation and waves) is given by the sum of the skin stress and the wave stress (Fig. 2), the latter of which represents the residual effect of both the normal stress and the tangential stress associated with the waves (not shown, cf. Fan et al., 2010; Donelan et al., 2012; Appendix B of AG12). The former (the skin stress) represents the direct transfer of momentum from wind to ocean circulation, and should match $\nu_4(\overline{\mathbf{V}}_2^{\varepsilon} - \mathbf{V}_2^{qs})_z$ at the base of the TVSBL ($z = \overline{\eta} - \delta$). The latter (the wave stress) represents the transfer of momentum from wind to waves. The momentum of waves is eventually transferred to circulation when/where waves are dissipated by the turbulent viscosity, as is shown by the last term of (49). To summarize $\nu_4(\overline{\mathbf{V}}_2^{\varepsilon} - \mathbf{V}_2^{qs})_z =$ [the skin stress] is the surface boundary condition applicable to numerical circulation models. The net momentum input to ocean circulation is given by the sum of the skin stress and the vertical integral of the last term of (49) associated with the wave dissipation. The net momentum input to traditional ocean circulation models (that is models that do not include wave effects) is given by the wind stress based, for example, on the Large and Pond (1981) parameterization, and should be compared with the sum of the skin stress and the depth integral of the wave dissipation term mentioned above.

5. Summary and discussion

The fundamentals of the vertically Lagrangian and horizontally Eulerian (VL) framework have been developed in the present study concerning the effect of surface waves on circulations in the upper ocean. We suggest that the thickness-weighted-mean (TWM) momentum equations of Mellor (2003), Broström et al. (2008), and AG12 correspond to the Lagrangian average of (1) which is the direct expression of the Lagrangian momentum equations. To our knowledge, no previous study has shown (i) how to derive the transformed expression for the TWM momentum equations corresponding to the Lagrangian average of (2) and (ii) how to introduce the concept of pseudomomentum to the VL framework, as can be seen in, for example, the discussion between Ardhuin et al. (2008a) and Mellor (2008b).

In Section 2 we have shown that the traditional pressure-based form stress term can be transformed into a set of terms that do not contain any pressure quantities. The transformation in the present study is applicable to a nonlinear equation system, which is an improvement over AG13 who utilised a version of the transformation based on a linear equation system for the waves. An important byproduct is that the velocity-based form stress term includes the time derivative of a wave-induced velocity which is referred to as the pseudomomentum in the VL framework, as it is analogous to the generalized pseudomomentum in AM78. The result is that the transformed expression of the TWM momentum equations (i.e. for the development of the quasi-EM velocity, namely the total transport velocity minus the VL pseudomomentum vector) has been obtained in Section 3. As shown in Appendix B, it is possible to derive the vortex force using the VL framework, using an approach that is a hybrid of Leibovich (1980) and CL76. We also noted that the twin expressions for the Lagrangian mean momentum equations (and hence also for the TWM momentum equations) may be traced back to the work of Lagrange (1788), which has been little mentioned in previous studies.

A nice feature of the VL framework is the treatment of the turbulent viscosity term near the sea surface. The traditional explanation for the viscosity-induced transfer of momentum from waves to circulation has been based on depth-integrated equations (e.g. Smith, 2006; Weber et al., 2006; Fan et al., 2010), whereas our explanation in Section 4 is based on depth-dependent equations with a vertically nonuniform viscosity coefficient, and thus is useful for revisiting the surface boundary condition used in numerical circulation models. We have shown that the velocity-based expression of the form stress term contains the residual effect of viscosity [see (48)]. In the transformed expression of the TWM momentum equations, the effect of viscosity appears as the sum of a flux-divergence term (which is associated with the skin stress applied by wind) and a body-force (which represents transfer of momentum from waves to circulation associated with the dissipation of wave kinetic energy) [see (49)]. This allows us to explain the concept of the virtual wave stress (VWS) of Longuet-Higgins (1953, 1960), without relying on the explicit use of the analytical solution of waves in the thin viscous boundary layer at the sea surface as in the work of Ünlüata and Mei (1970), Xu and Bowen (1994), Ng (2004) and AG12. Our explanation may be regarded as a recipe for a future study to reexplain the VWS using the three-dimensional Lagrangian framework of AM78, which has not been achieved in previous studies despite the utility of AM78 to allow a general spectrum of waves (cf. Ardhuin et al., 2008b).

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A. Analogy to Andrews and McIntyre (1978)

The vector (Ξ_1, Ξ_2, Ξ_3) in AM78 represents the position of a fluid particle in the Eulerian-Cartesian coordinates,[A1] corresponding to $(x^\varepsilon, y^\varepsilon, z^\varepsilon)$ in the present study. Likewise (x_1, x_2, x_3) in AM78 represents the position of a fluid particle in the three-dimensional Lagrangian coordinates,[A2] corresponding to (x, y, z) in the present study. Thus the fluctuation of the position $(\xi_1, \xi_2, \xi_3) \equiv (\Xi_1, \Xi_2, \Xi_3) - (x_1, x_2, x_3)$ in AM78 corresponds to $(0, 0, z''')$ in the present study. Noting that $\xi_j = \Xi_j - x_j$ and using the notation of AM78, their equations (B.1)-(B.4) may be rewritten,

$$\begin{aligned}
\xi_{j,i}\overline{D}^L(u_j^\xi) &= \overline{D}^L(\xi_{j,i}u_j^\xi) - u_j^\xi\{(\overline{D}^L\xi_j)_{,i} - \overline{u}^L_{k,i}\xi_{j,k}\} \\
&= \overline{D}^L(\xi_{j,i}u_j^\xi) - (\overline{u}_j^L + u_j^l)u^l_{j,i} + \overline{u}^L_{k,i}\xi_{j,k}u_j^\xi, \qquad \text{(A1a)} \\
\mathsf{p}_i &\equiv -\overline{\xi_{j,i}u_j^l}^L = -\overline{\xi_{j,i}u_j^\xi}^L, \qquad \text{(A1b)} \\
-\overline{\xi_{j,i}\overline{D}^L(u_j^\xi)}^L &= \overline{D}^L(\mathsf{p}_i) + \overline{u_j^l(u^l_{j,i})}^L + \overline{u}^L_{k,i}(\mathsf{p}_k), \qquad \text{(A1c)}
\end{aligned}$$

where $\overline{D}^L\xi_j = u_j^l = u_j^\xi - \overline{u}_j^L$ is understood. The quantity p_i is the generalized pseudomomentum for waves in a non-rotating frame (AM78). The first term on the rhs of (A1c) is analogous to $-\widehat{\mathfrak{D}}_t(\overline{z'''_X w'''})$ in (20). The second term of (A1c) is analogous to $+\overline{w'''_X w'''}$ in (20). The third term of (A1c) is analogous to $-\widehat{\mathbf{V}}_X \cdot \overline{(\nabla z''')w'''}$ in (20).

B. The derivation of the vortex force

In Section 3c, the TWM momentum equations are rewritten for the development of the equivalent of the quasi-EM velocity, namely the total transport velocity minus the VL pseudomomentum vector.

[A1]Numeric subscripts in AM78 represent axes in Cartesian coordinates, and should not be confused with numeric subscripts in the present study representing the order of an asymptotic expansion.

[A2]This has been written (a, b, c) in Section 1 of the present study. In AM78, the coordinate transformation matrix has been written as $\Xi_{j,i} \equiv \partial\Xi_j/\partial x_i$.

The last term of each of (36a)-(36c) is (the legacy of) the lateral Reynolds stress term, which can be expanded using the recipe of CL76, as shown below.

a. *The proto-type vortex force equations*

In order to manipulate (36a)-(36c), we use several identities derived from the condition of horizontally homogeneous waves (29b). The first identity is

$$\nabla\widehat{A}_2 = \nabla\overline{A}_2 = \nabla\overline{A}_2^{\varepsilon}, \tag{B1}$$

where A_2 is an arbitrary quantity at $O(\alpha^2)$, and (31a)-(31b) have been used. The second identity is that the vertical component of the quasi-Stokes velocity is zero, as shown by (32b), which leads to

$$\begin{aligned}\widehat{\mathcal{D}}_T &= \overline{\mathcal{D}}_T^{\varepsilon} + \mathbf{V}_2^{qs}\cdot\nabla + w_2^{qs}\partial_z \\ &= \overline{\mathcal{D}}_T^{\varepsilon} + \mathbf{V}_2^{qs}\cdot\nabla,\end{aligned} \tag{B2}$$

where $\overline{\mathcal{D}}_T^{\varepsilon} \equiv \partial_T + \overline{\mathbf{V}}^{\varepsilon}\cdot\nabla + \overline{w}^{\varepsilon}\partial_z$ is the material derivative operator based on the EM velocity. The third identity is

$$-\overline{(\nabla z_1''')w_1'''} = \overline{z_1'''\mathbf{V}_{1z}'''}, \tag{B3}$$

which has been derived using (29b) and (34), and is the relationship found in the second term on the rhs of (38a). Using (B1)-(B3), we rewrite (36a)-(36c) as

$$\overline{\mathcal{D}}_T^{\varepsilon}\overline{\mathbf{V}}_2^{\varepsilon} = -\nabla(\overline{p}_4 + \overline{\eta}_4 + \overline{\pi}_4) - (\nabla\times\overline{\mathbf{V}}_2^{\varepsilon})\times\overline{z_1'''\mathbf{V}_{1z}'''} - (\overline{\mathbf{V}'''\cdot\nabla\mathbf{V}'''})_4, \tag{B4a}$$

$$\begin{aligned}\overline{\mathcal{D}}_T^{\varepsilon}\overline{w}_2^{\varepsilon} &= -\partial_z(\overline{p}_4 + \overline{\pi}_4) - \mathbf{V}_2^{qs}\cdot\nabla\overline{w}_2^{\varepsilon} + \widehat{\mathbf{V}}_{2z}\cdot\overline{z_1'''\mathbf{V}_{1z}'''} \\ &\quad + [\overline{\mathbf{V}''\cdot(w_z'''\nabla z''' - z_z'''\nabla w''')}]_4 - (\overline{\mathbf{V}'''\cdot\nabla w'''})_4.\end{aligned} \tag{B4b}$$

These equations contain a prototype of the vortex force.[B1] The analysis up to this point was sufficient for Leibovich (1980) to derive the vortex force. This is because he used the three-dimensional Lagrangian framework of AM78. By contrast we still have the last term of each of (B4a)-(B4b) to work on further, because the VL framework is Eulerian in the horizontal direction. In order to deal with these terms, we use the fact that the lateral advection of velocity in the VL coordinates can be written as

$$\overline{\mathbf{V}''' \cdot \nabla \mathbf{V}'''} = \frac{1}{2}\nabla\overline{|\mathbf{V}'''|^2} + \overline{(\nabla \times \mathbf{V}''') \times \mathbf{V}'''}, \tag{B5a}$$

$$\overline{\mathbf{V}''' \cdot \nabla w'''} = \frac{1}{2}\partial_z\overline{|\mathbf{V}'''|^2} + \overline{(\nabla w''' - \mathbf{V}_z''') \cdot \mathbf{V}'''}, \tag{B5b}$$

where here $\nabla \times \mathbf{V}''' = (v_x''' - u_y''')\mathbf{z}$, and manipulate the last term of (B5a)-(B5b) using the approach of CL76, as shown below.

b. *Vorticity equations: application of Craik and Leibovich (1976)*

First we rewrite the momentum equations (11a)-(11b) in the VL coordinates as

$$\mathbf{V}_t + (\nabla \times \mathbf{V}) \times \mathbf{V} + \varpi \mathbf{V}_z = -\nabla(p + \eta + |\mathbf{V}|^2/2) - (\mathcal{D}_t w)\nabla z''', \tag{B6a}$$

$$w_t + (\nabla w - \mathbf{V}_z) \cdot \mathbf{V} + \varpi w_z = -\partial_z(p + |\mathbf{V}|^2/2) - (\mathcal{D}_t w) z_z''', \tag{B6b}$$

where another version of (B5a)-(B5b) has been used to rewrite the advection terms. Because both $O(\alpha)$ and $O(\alpha^2)$ wave motions satisfy the apparent irrotational condition (34) and ϖ'' is sufficiently small, momentum equations for $O(\alpha^3)$ waves may be written,

$$\mathbf{V}_{3\tau}''' + (\nabla \times \overline{\mathbf{V}}_2) \times \mathbf{V}_1''' + \widehat{\varpi}_2 \mathbf{V}_{1z}''' = -\nabla(p + \eta + |\mathbf{V}|^2/2)_3''' - [(\mathcal{D}_t w)\nabla z''']_3''', \tag{B7a}$$

$$w_{3\tau}''' + (\nabla \overline{w}_2 - \overline{\mathbf{V}}_{2z}) \cdot \mathbf{V}_1''' + \widehat{\varpi}_2 w_{1z}''' = -\partial_z(p + |\mathbf{V}|^2/2)_3''' - [(\mathcal{D}_t w) z_z''']_3'''. \tag{B7b}$$

[B1]Note that in (B4a), the cross-product operator is the vector invariant cross-product. This is because for low-pass filtered quantities, the VL coordinates correspond to the standard Eulerian-Cartesian coordinates as pointed out by Jacobson and Aiki (2006).

Cross-derivative of the above equations yields

$$(\nabla \times \mathbf{V}_3''')_\tau + \underbrace{\nabla \phi_{1\tau}'''}_{\cos\theta} \cdot \nabla(\nabla \times \overline{\mathbf{V}}_2) - (\nabla \times \overline{\mathbf{V}}_2)\underbrace{\phi_{1zz\tau}'''}_{\sin\theta} + \nabla\widehat{\varpi}_2 \times \underbrace{\nabla\phi_{1z\tau}'''}_{\cos\theta} = -\underbrace{[\nabla(\mathcal{D}_t w) \times \nabla z''']_3'''}_{0}, \tag{B8a}$$

$$\begin{aligned}(w_{3x}''' - u_{3z}''')_\tau + [(\nabla \times \overline{\mathbf{V}}_2)\underbrace{\phi_{1y\tau}'''}_{\cos\theta}]_z + (\nabla\overline{w}_2 - \overline{\mathbf{V}}_{2z})_x \cdot \underbrace{\nabla\phi_{1\tau}'''}_{\cos\theta} + (\nabla\overline{w}_2 - \overline{\mathbf{V}}_{2z}) \cdot \underbrace{\nabla\phi_{1x\tau}'''}_{\sin\theta} \\ + \widehat{\varpi}_{2x}\underbrace{\phi_{1zz\tau}'''}_{\sin\theta} - \widehat{\varpi}_{2z}\underbrace{\phi_{1xz\tau}'''}_{\cos\theta} = \underbrace{[(\mathcal{D}_t w)_z z_x''' - (\mathcal{D}_t w)_x z_z''']_3'''}_{\sin\theta},\end{aligned} \tag{B8b}$$

$$\begin{aligned}(w_{3y}''' - v_{3z}''')_\tau - [(\nabla \times \overline{\mathbf{V}}_2)\underbrace{\phi_{1x\tau}'''}_{\cos\theta}]_z + (\nabla\overline{w}_2 - \overline{\mathbf{V}}_{2z})_y \cdot \underbrace{\nabla\phi_{1\tau}'''}_{\cos\theta} + (\nabla\overline{w}_2 - \overline{\mathbf{V}}_{2z}) \cdot \underbrace{\nabla\phi_{1y\tau}'''}_{\sin\theta} \\ + \widehat{\varpi}_{2y}\underbrace{\phi_{1zz\tau}'''}_{\sin\theta} - \widehat{\varpi}_{2z}\underbrace{\phi_{1yz\tau}'''}_{\cos\theta} = \underbrace{[(\mathcal{D}_t w)_z z_y''' - (\mathcal{D}_t w)_y z_z''']_3'''}_{\sin\theta},\end{aligned} \tag{B8c}$$

where $(\mathbf{V}_1''', w_1''') = (\nabla\phi_{1\tau}''', \phi_{1z\tau}''')$ and $\nabla^2\phi_1''' + \phi_{1zz}''' = 0$ have been used. The rhs of (B8a) vanishes because both $O(\alpha)$ and $O(\alpha)$ waves are proportional to $\nabla\theta = (k.l)$ (a similar discussion appears in footnote 8).

We now use again the fact that $O(\alpha)$ and $O(\alpha^2)$ waves satisfy the apparent irrotational condition (34) to write the last term of (B5a)-(B5b) at $O(\alpha^4)$ as

$$\begin{aligned}[\overline{(\nabla \times \mathbf{V}''') \times \mathbf{V}'''}]_4 &= \overline{(\nabla \times \mathbf{V}_3''') \times \mathbf{V}_1'''} \\ &= -\overline{(\nabla \times \mathbf{V}_3''')_\tau \times \nabla\phi_1'''} \\ &= -(\nabla \times \overline{\mathbf{V}}_2) \times \overline{\phi_{1zz\tau}'''\nabla\phi_1'''} \\ &= (\nabla \times \overline{\mathbf{V}}_2^{\varepsilon}) \times \overline{z_{1z}'''\mathbf{V}_1'''},\end{aligned} \tag{B9a}$$

where the second line has been derived by first substituting for $(\nabla \times \mathbf{V}_3''')_\tau$ using (B8a) and then retaining terms labelled $\sin\theta$ in (B8a) because $\nabla\phi'''_1$ is proportional to $\sin\theta$, and

$$
\begin{aligned}
[\overline{(\nabla w''' - \mathbf{V}'''_z)\cdot \mathbf{V}'''}]_4 &= \overline{(\nabla w'''_3 - \mathbf{V}'''_{3z})\cdot\mathbf{V}'''_1} \\
&= -\overline{(\nabla w'''_3 - \mathbf{V}'''_{3z})_\tau \cdot \nabla\phi'''_1} \\
&= -(\nabla\overline{w}_2 - \overline{\mathbf{V}}_{2z})\cdot \overline{(\nabla\phi'''_{1\tau})(\phi'''_{1xx}+\phi'''_{1yy})} \\
&\quad +\nabla\widehat{\varpi}_2\cdot\overline{(\nabla\phi'''_1)\phi'''_{1zz\tau}} \\
&\quad -\overline{[(\mathcal{D}_t w)_z \nabla z''' - z'''_z\nabla(\mathcal{D}_t w)]_3\cdot\nabla\phi'''_1} \\
&= -\overline{\mathbf{V}}_{2z}\cdot\overline{z'''_{1z}\mathbf{V}'''_1} \\
&\quad -\overline{[(\mathcal{D}_t w)_z \nabla z''' - z'''_z\nabla(\mathcal{D}_t w)]_3\cdot\nabla\phi'''_1}, \qquad \text{(B9b)}
\end{aligned}
$$

where the second line has been derived by first substituting for $(\nabla w'''_3 - \mathbf{V}'''_{3z})_\tau$ using (B8b)-(B8c) and then retaining terms labelled $\sin\theta$ in (B8b)-(B8c) because $\nabla\phi'''_1$ is proportional to $\sin\theta$. The above procedure is based on CL76. The last line of (B9b) has been derived using $\nabla\overline{w}_2 = \nabla\widehat{\varpi}_2$ which follows from (17) and (B1).

Substitution of (B9a)-(B9b) to (B4a)-(B4b), using (B5a)-(B5b), then yields

$$
\overline{\mathcal{D}}^{\varepsilon}_T \overline{\mathbf{V}}^{\varepsilon}_2 = -\nabla(\overline{p}_4+\overline{\eta}_4+\overline{\Pi}_4) - (\nabla\times\overline{\mathbf{V}}^{\varepsilon}_2)\times\mathbf{V}^{qs}_2, \qquad \text{(B10a)}
$$

$$
\begin{aligned}
\overline{\mathcal{D}}^{\varepsilon}_T \overline{w}^{\varepsilon}_2 &= -\partial_z(\overline{p}_4+\overline{\Pi}_4) - (\nabla\overline{w}^{\varepsilon}_2 - \overline{\mathbf{V}}^{\varepsilon}_{2z})\cdot\mathbf{V}^{qs}_2 \\
&\quad +\frac{1}{2}\partial_z(|\mathbf{V}^{qs}_2|^2 - |\overline{z'''_{1z}\mathbf{V}'''_1}|^2) \\
&\quad +[\overline{\mathbf{V}''\cdot(w'''_z\nabla z''' - z'''_z\nabla w''')}]_4 \\
&\quad +\overline{\nabla\phi'''_1\cdot[(\mathcal{D}_t w)'''_z\nabla z''' - z'''_z\nabla(\mathcal{D}_t w)''']_3}, \qquad \text{(B10b)}
\end{aligned}
$$

where the third last term of (B10b) has been derived using (31a)-(32a),[B2] and

$$
\overline{\Pi}_4 \equiv \frac{1}{2}(\overline{|\mathbf{V}'''|^2 - w'''^2})_4 + [\overline{(\mathbf{V}''\cdot\nabla z''')w'''}]_4, \qquad \text{(B10c)}
$$

[B2] $(\widehat{\mathbf{V}}_2 - \overline{\mathbf{V}}^{\varepsilon}_2)_z\cdot(\overline{z'''_1\mathbf{V}'''_{1z}}) + (\overline{\mathbf{V}}_2 - \overline{\mathbf{V}}^{\varepsilon}_2)_z\cdot(\overline{z'''_{1z}\mathbf{V}'''_1}) = \mathbf{V}^{qs}_{2z}\cdot\mathbf{V}^{qs}_2 - (\overline{z'''_{1z}\mathbf{V}'''_1})_z\cdot(\overline{z'''_{1z}\mathbf{V}'''_1})$.

is the revised Bernoulli head. The last three terms of (B10b) consist of waves up to $O(\alpha^2)$. Because $O(\alpha)$ and $O(\alpha^2)$ waves are horizontally homogeneous, these three terms can be absorbed to the Bernoulli head in (B10c) without affecting the horizontal component of wave-averaged momentum equation (B10a). Looking at (B10a)-(B10c), $O(\alpha^3)$ waves appear only in the first term of the Bernoulli head (B10c).

C. Discussion on the different scaling for the variation of circulations

The utility of the velocity-based expression of the form stress term (as well as the pseudomomentum in the VL framework) is not limited to the scaling of LCs in CL76. There are various choices for the scaling of the temporal and horizontal variations of circulation, as argued in Lane et al. (2007) and AG13, and are briefly explained in this section using a generalized expression for the scaling. Let consider circulations whose time development is m orders (in terms of α, where $m = 0, 1, 2, ...$) slower than the phase cycle of waves, and the horizontal scale of circulations is n orders (in terms of α, where $n = 0, 1, 2, ...$) larger than wavelength. The time derivative operator may be decomposed as $\partial_t = \partial_\tau + \alpha^m \partial_T$ where ∂_τ operates on wave quantities and ∂_T operates on the low-pass filtered quantities (i.e. circulations as well as the slow time evolution of the wave quantities). Likewise the lateral gradient operator may be decomposed as $\nabla = \dot{\nabla} + \alpha^n \overline{\nabla}$ where $\dot{\nabla}$ operates on wave quantities and $\overline{\nabla}$ operates on the low-pass filtered quantities (i.e. circulations as well as on the large spatial-scale variation of the wave quantities).[C1] To summarize, $\partial_\tau \overline{A} = 0$ and $\partial_T \overline{A} \neq 0$ (this is as in Section 3), likewise $\dot{\nabla}\overline{A} = 0$ and $\overline{\nabla}\,\overline{A} \neq 0$ for an arbitrary quantity

[C1] Note that $\partial_t A''' = \partial_\tau A''' + \alpha^m \partial_T A'''$ and $\partial_t \overline{A} = \alpha^m \partial_T \overline{A}$ for arbitrary wave and mean quantities A''' and $\overline{A}$. Likewise $\nabla A''' = \dot{\nabla} A''' + \alpha^n \overline{\nabla} A'''$ and $\nabla \overline{A} = \alpha^n \overline{\nabla}\,\overline{A}$. It should be also noted that the amplitude, wavenumber, and frequency of $O(\alpha)$ waves are constant on the time and horizontal scales of waves (i.e. $\partial_\tau A = 0$ and $\dot{\nabla} A = 0$ for $A = \mathcal{A},\ k,\ l,\ \sigma$) but may vary on the time and horizontal scales of low-pass filtered quantities (i.e. $\partial_T A \neq 0$ and $\overline{\nabla} A \neq 0$ for $A = \mathcal{A},\ k,\ l,\ \sigma$). These rules and notations are the same as that in AG13.

A, leading to

$$\partial_\tau(\overline{A_1'''B_1'''}) = 0, \quad \text{and} \quad \partial_T(\overline{A_1'''B_1'''}) \neq 0, \tag{C1a}$$

$$\dot{\nabla}(\overline{A_1'''B_1'''}) = 0, \quad \text{and} \quad \overline{\nabla}(\overline{A_1'''B_1'''}) \neq 0, \tag{C1b}$$

where A_1''' and B_1''' are arbitrary quantities at $O(\alpha)$.

The set of wave-averaged momentum equations that contain the vortex force may be derived if $m = n+2$. Substitution of $n = 0$ recovers the scaling for the low-pass filtered flow used in Section 3 and is the same scaling as used in CL76. Substitution of $n = 2$ recovers the scaling of circulation in an inner coastal shelf region in McWilliams et al. (2004). The TWM equation system (6a)-(6c) is written as

$$\overline{\nabla} \cdot \widehat{\mathbf{V}}_2 + \partial_z \widehat{\varpi}_{n+2} = 0, \tag{C2a}$$

$$(\partial_T + \widehat{\mathbf{V}}_2 \cdot \overline{\nabla} + \widehat{\varpi}_{n+2}\partial_z)\widehat{\mathbf{V}}_2 + \mathcal{RS}^{\mathbf{V}}_{n+4} = -\overline{\nabla}(\overline{p}_4 + \overline{\eta}_4) + \mathcal{FS}^{\mathbf{V}}_{n+4}, \tag{C2b}$$

$$\underbrace{(\partial_T + \widehat{\mathbf{V}}_2 \cdot \overline{\nabla} + \widehat{\varpi}_2\partial_z)\widehat{w}_2}_{\text{present only when n=0}} + \mathcal{RS}^{w}_{4} = -\overline{p}_{4z}, \tag{C2c}$$

where the material derivative term in (C2c) is present only when $n = 0$. The horizontal and vertical momentum equations have been written at $O(\alpha^{n+4})$ and $O(\alpha^4)$, respectively. Using the recipe of the present study, the Reynolds stress term and the form stress term in the horizontal momentum equation (C2b) can be transformed to the horizontal component of the vortex force, $\mathbf{V}_2^{qs} \times (\overline{\nabla} \times \overline{\mathbf{V}}_2^{\varepsilon})$, or its variant which is $O(\alpha^{n+4})$. Likewise the Reynolds stress term in the vertical momentum equation (C2c) can be transformed to the vertical component of the vortex force, $\mathbf{V}_2^{qs} \cdot (\partial_z \overline{\mathbf{V}}_2^{\varepsilon} - \overline{\nabla}\overline{w}_2^{\varepsilon})$, or its variant which is $O(\alpha^4)$. Another consequence is that the TWM momentum equations (C2b)-(C2c) are to be rewritten for the development of the EM velocity.

An alternative and classical form of the wave-averaged momentum equations includes the so-called radiation stress (e.g. Longuet-Higgins and Stewart, 1964, hereafter LHS64; Bühler and Jacobson, 2001; Mellor, 2003). The depth-integrated radiation stress of LHS64 has been written by these authors as the product of $O(\alpha)$ wave quantities. The sum of the Reynolds stress term (7) and (the negative of) the form stress term (8) then appears at $O(\alpha^{n+2})$ and when integrated over the water depth reads,

$$\begin{aligned}\int_{-\infty}^{\overline{\eta}} [\mathcal{RS}^{\mathbf{V}}_{n+2} - \mathcal{FS}^{\mathbf{V}}_{n+2}]\, dz &= \overline{\nabla} \cdot \int_{-\infty}^{\overline{\eta}} \overline{\mathbf{V}_1''' \mathbf{V}_1'''}\, dz + \frac{1}{2}\overline{\nabla}\,\overline{\eta_1'''^2} - \overline{\nabla} \int_{-\infty}^{\overline{\eta}} (\overline{z_1''' p_{1z}'''}) dz \\ &= \underbrace{\overline{\nabla} \cdot \int_{-\infty}^{\overline{\eta}} \overline{\mathbf{V}_1''' \mathbf{V}_1'''}\, dz}_{S_{xx}^{(1)}} + \underbrace{\overline{\nabla} \int_{-\infty}^{\overline{\eta}} (\overline{-w_1'''^2})\, dz}_{S_{xx}^{(2)}} + \overline{\nabla}\underbrace{\frac{1}{2}\overline{\eta_1'''^2}}_{S_{xx}^{(3)}}, \end{aligned} \tag{C3}$$

where $\overline{z_1''' p_{1z}'''} = -\overline{z_1''' w_{1\tau}'''} = \overline{z_{1\tau}''' w_1'''}$ has been used following (C1a), and $S_{xx}^{(1)}$, $S_{xx}^{(2)}$, and $S_{xx}^{(3)}$ are the notation in LHS64. The stress terms, $\mathcal{RS}^{\mathbf{V}}_{n+2}$ and $\mathcal{FS}^{\mathbf{V}}_{n+2}$, are part of the horizontal component of the TWM momentum equations written at $O(\alpha^{n+2})$. In order for the tendency term of the wave-averaged momentum equations to be written at $O(\alpha^{n+2})$, the time derivative operator needs to be decomposed as $\partial_t = \partial_\tau + \alpha^n \partial_T$ which means that $m = n$. Namely LHS64 consider circulations whose time development is n orders slower (in term of α) than the phase cycle of the waves, which is two orders faster than that in the previous paragraph (i.e. the vortex force regime). The TWM equation system (6a)-(6c) becomes

$$\overline{\nabla} \cdot \widehat{\mathbf{V}}_2 + \partial_z \widehat{\varpi}_{n+2} = 0, \tag{C4a}$$

$$\partial_T \widehat{\mathbf{V}}_2 + \mathcal{RS}^{\mathbf{V}}_{n+2} = -\overline{\nabla}(\overline{p}_2 + \overline{\eta}_2) + \mathcal{FS}^{\mathbf{V}}_{n+2}, \tag{C4b}$$

$$0 = -\overline{p}_{2z}. \tag{C4c}$$

AG13 have specialized to the case of $n = 1$ (but the result holds for $n = 2, 3, ..$) and show that the depth-dependent radiation stress term is rewritten as

$$\mathcal{RS}^{\mathbf{V}}_{n+2} - \mathcal{FS}^{\mathbf{V}}_{n+2} = -\partial_T \mathbf{V}_2^{qs} + \overline{\nabla}\frac{1}{2}(\overline{|\mathbf{V}_1'''|^2 - w_1'''^2}), \tag{C5}$$

which contains no singular treatment at the sea surface, in contrast to Mellor (2008a). Substitution of (C5) to (C4b) yields a wave-averaged momentum equation written for the development of the EM velocity. The last term of (C5) vanishes in the present study because of the use of deep water waves, however the term has been kept in order for readers to see correspondence to $\widehat{\zeta}$ in McWilliams et al. (2004), J in Smith (2006), and S^J in Ardhuin et al. (2008b).

The difference of the incompressibility conditions of the total transport velocity, (C2a) or (C4a), and the EM velocity, $\overline{\nabla} \cdot \overline{\mathbf{V}}_2^{\varepsilon} + \partial_z \overline{w}_{n+2}^{\varepsilon} = 0$, yields

$$\overline{\nabla} \cdot \mathbf{V}_2^{qs} + \partial_z w_{n+2}^{qs} = 0, \tag{C6}$$

which indicates that, in the presence of the slow horizontal variations of waves, the vertical component of the quasi-Stokes velocity is nonzero and scaled at $O(\alpha^{n+2})$ (cf. Tamura et al., 2012).

D. Viscosity term in the energy equation

The last line of (47) is the product between $-(\dot{\nabla}\theta)/\sigma$ and the viscosity term of the depth-dependent energy equation as we now show. As noted by Phillips (1977), Weber et al. (2006), and AG12, the viscosity term of the depth-dependent energy equation may be separated into two terms: one is the vertical divergence of a viscosity-induced flux (noted as FluxDiv) and one is dissipation at depths excluding the thin viscous boundary layers at the sea surface (noted as Dissipation). These two terms can be obtained by substituting (43a)-(43c) into the FluxDiv-Dissipation part of (47) to give

$$
\begin{aligned}
\mathsf{FluxDiv} &= [\nu_4 \overline{\mathbf{V}_1''' \cdot (\dot{\nabla} w_1''' + \mathbf{V}_{1z}''')} + 2\nu_4 \overline{w_1''' w_{1z}'''}]_z \\
&= [\nu_4 (\overline{|\mathbf{V}_1'''|^2 + w_1'''^2})_z]_z \\
&= \sigma^2 [\nu_4 (\overline{\kappa^2 \phi_1'''^2 + \phi_{1z\theta}'''^2})_z]_z \\
&= \sigma^2 [\nu_4 (\overline{\kappa^2 \phi_1'''^2 + \phi_{1z}'''^2})_z]_z \\
&= \sigma^2 [\nu_4 (\overline{\phi_1''' \phi_{1z}'''})_{zz}]_z, & \text{(D1a)} \\
\mathsf{Dissipation} &= 2\nu_4 [\overline{u_{1x}'''^2 + v_{1y}'''^2 + w_{1z}'''^2}] + \nu_4 [\overline{(v_{1x}''' + u_{1y}''')^2} + \overline{(w_{1x}''' + u_{1z}''')^2} + \overline{(w_{1y}''' + v_{1z}''')^2}] \\
&= 2\nu_4 [\overline{u_{1x}'''^2 + v_{1y}'''^2 + w_{1z}'''^2}] + 4\nu_4 [\overline{u_{1y}'''^2 + v_{1z}'''^2 + w_{1x}'''^2}] \\
&= 2\nu_4 \sigma^2 [\overline{(k^4 + l^4) \phi_{1\theta}'''^2 + \kappa^4 \phi_{1\theta}'''^2}] + 4\nu_4 \sigma^2 [\overline{(kl)^2 \phi_{1\theta}'''^2 + \kappa^2 \phi_{1z}'''^2}] \\
&= 4\kappa^2 \nu_4 \sigma^2 [\overline{\kappa^2 \phi_{1\theta}'''^2 + \phi_{1z}'''^2}] \\
&= \nu_4 \sigma^2 (\overline{\phi_1''' \phi_{1z}'''})_{zzz}, & \text{(D1b)}
\end{aligned}
$$

where the second line of each equation has been derived using (34),[D1] and the third line of each equation has been derived using (30c)-(30d). It should be noted that to connect to (49), use is made of (32a).

The above two terms are part of the depth-dependent wave energy equation, which may be derived

[D1] The expression of the dissipation rate as given by the last line of (D1b) is associated with only irrotational wave motions in the vertical plane, in particular at depths below the base of the TVSBL. In the present study (Section 4), the coefficient of turbulent viscosity (or $\nu\kappa^2/\sigma$) has been scaled at $O(\alpha^4)$ and thus the dissipation rate inside the TVSBL can be neglected, which follows Phillips (1977, page 52) and Appendix B of AG12. However, if we consider breaking waves under high wind conditions, the coefficient of turbulent viscosity (or $\nu\kappa^2/\sigma$) might be scaled at $O(\alpha^2)$ or $O(\alpha^3)$ in the vicinity of the sea surface (Drazen et al., 2008; Tian et al., 2010). Then the dissipation rate associated with rotational wave motions inside the TVSBL (which is no longer thin) should be retained by revisiting (43a)-(43c) and (D1a)-(D1b), a topic we shall discuss in a later paper.

taking the sum of Eqs. (A6a) and (A6b) of AG12 and then picking-up $O(\alpha^6)$ terms to yield,

$$\frac{1}{2}\partial_T(\overline{|\mathbf{V}_1'''|^2 + w_1'''^2}) + \partial_z[\overline{z_{1T}'''(p_1''' + \eta_1''')}] + \overline{\nabla} \cdot [\overline{\mathbf{V}_1'''(p_1''' + \eta_1''')}]$$
$$= -\partial_z[\overline{z_\tau'''(p''' + \eta''')}]_6 + \mathsf{FluxDiv} - \mathsf{Dissipation}, \qquad \text{(D2)}$$

which indicates that the vertical flux of energy is given by the first two terms on the rhs, one is induced by pressure and one is induced by viscosity. Thus, if we consider the vertical integral of (D2) from $z = -\infty$ to $z = \overline{\eta} - \delta$ (where δ is the thickness of the TVSBL), the source of wave energy is given by the combined vertical flux $-[\overline{z_\tau'''(p''' + \eta''')}]_6 + \sigma^2\nu_4(\overline{\phi_1'''\phi_{1z}'''})_{zz}$ evaluated at $z = \overline{\eta} - \delta$. When there is no wind forcing, we obtain $[\overline{z_\tau'''(p''' + \eta''')}]_6 = \sigma^2\nu_4(\overline{\phi_1'''\phi_{1z}'''})_{zz}$ at $z = \overline{\eta} - \delta$, with a consequence that the vertical integral of the first two terms on the rhs of (D2) cancel each other. Wave energy is then gradually decreased by the last term of (D2). Nevertheless the viscosity-induced momentum flux, $\nu_4\mathbf{V}_{2z}^{qs}$, at $z = \overline{\eta} - \delta$ in (49) is nonzero, so that it remains to control the boundary condition of the EM velocity in (49).

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List of Tables

Table 1: Classification of previous studies based on the form of the Lagrangian momentum equations that is used.

Direct Expression	Transformed Expression
Lagrange (1788, Eq. C on page 445)	Lagrange (1788, Eq. D on page 446)
	Lamb (1932, 2nd Eq. on page 13)
Pierson (1962, Eqs. 5, 9)	Pierson (1962, Eqs. 4, 10)
Andrews & McIntyre (1978, Eq. 8.7a)	Andrews & McIntyre (1978, Eq. 3.8)
Mellor (2003), Aiki & Greatbatch (2012)	Aiki & Greatbatch (2013, simplified), this study (nonlinear)

Table 2: List of symbols, where A is an arbitrary quantity. $(x^\varepsilon, y^\varepsilon, z^\varepsilon, t^\varepsilon)$ and ϖ are the same as (x^c, y^c, z^c, t^c) and w^* in Jacobson and Aiki (2006), AG12, and AG13.

$(x^\varepsilon, y^\varepsilon, z^\varepsilon)$	Eulerian-Cartesian coordinates
(a, b, c)	Three-dimensional Lagrangian coordinates
(x, y, z)	Vertically Lagrangian and horizontally Eulerian (VL) coordinates
$\overline{A}^\varepsilon$	Time-mean in Eulerian-Cartesian coordinates
$\widehat{A} \equiv \overline{z_z^\varepsilon A}$	Thickness-weighted time-mean in the VL coordinates
$\overline{A}$	Unweighted time-mean in the VL coordinates
$A' \equiv A - \overline{A}^\varepsilon$	Deviation from the Eulerian mean, compared at fixed z^ε ($\overline{A'}^\varepsilon = 0$)
$A'' \equiv A - \widehat{A}$	Deviation from the thickness-weighted mean, compared at fixed z ($\overline{z_z^\varepsilon A''} = 0$)
$A''' \equiv A - \overline{A}$	Deviation from the unweighted mean, compared at fixed z ($\overline{A'''} = 0$)
$\nabla \equiv (\partial_x, \partial_y)$	Lateral gradient in the VL coordinates ($\nabla z = 0, \nabla z^\varepsilon = \nabla z'''$)
$\nabla^\varepsilon \equiv (\partial_{x^\varepsilon}, \partial_{y^\varepsilon})$	Horizontal gradient in Eulerian-Cartesian coordinates ($\nabla^\varepsilon = \nabla - (\nabla z^\varepsilon)\partial_{z^\varepsilon}$)
$\mathbf{V} \equiv (u, v)$	Horizontal component of velocity
w	Vertical component of velocity
$\varpi \equiv (w - z_t^\varepsilon - \mathbf{V} \cdot \nabla z^\varepsilon)/z_z^\varepsilon$	Vertical velocity associated with volume flux through surface of fixed z
$(\widehat{\mathbf{V}}, \widehat{w})$	Thickness-weighted-mean (TWM) velocity
$(\widehat{\mathbf{V}}, \widehat{\varpi})$	Total transport velocity ($\nabla \cdot \widehat{\mathbf{V}} + \widehat{\varpi}_z = 0$)
$(\mathbf{V}^{qs}, w^{qs}) \equiv (\widehat{\mathbf{V}} - \overline{\mathbf{V}}^\varepsilon, \widehat{\varpi} - \overline{w}^\varepsilon)$	Quasi-Stokes velocity ($\nabla \cdot \mathbf{V}^{qs} + w_z^{qs} = 0$)
η	Sea surface height
p	Sum of oceanic nonhydrostatic pressure and atmospheric sea surface pressure
$\mathcal{FS}^{\mathbf{V}}$	Divergence of form stress $\equiv -[\overline{z'''\nabla(p''' + \eta''')}]_z + \nabla(\overline{z''' p_z'''})$
$\mathcal{RS}^A$ for $A = u, v$ and w	Divergence of the Reynolds stress $\equiv \nabla \cdot (\overline{z_z^\varepsilon \mathbf{V}'' A''}) + (\overline{z_z^\varepsilon \varpi'' A''})_z$
$\mathcal{D}_t \equiv \partial_t + \mathbf{V} \cdot \nabla + \varpi \partial_z$	Introduced in (11a)-(11b)
$\widehat{\mathcal{D}}_t \equiv \partial_t + \widehat{\mathbf{V}} \cdot \nabla + \widehat{\varpi} \partial_z$	Introduced in (9a)-(9b)
$\widehat{\mathcal{D}}_T \equiv \partial_T + \widehat{\mathbf{V}}_2 \cdot \nabla + \widehat{\varpi}_2 \partial_z$	Introduced in (23b)-(23c)
$\overline{\mathcal{D}}_T^\varepsilon \equiv \partial_T + \overline{\mathbf{V}}_2^\varepsilon \cdot \nabla + \overline{w}_2^\varepsilon \partial_z$	Introduced in (B2)
$\mathcal{A}$	Amplitude of $O(\alpha)$ wave
α	Surface slope of $O(\alpha)$ wave
$\kappa \equiv \sqrt{k^2 + l^2}$	Horizontal wavenumber of $O(\alpha)$ wave
σ	Frequency of $O(\alpha)$ wave
$\theta \equiv kx + ly - \sigma\tau$	Phase of $O(\alpha)$ wave
∂_τ	Time derivative operator for wave quantities
∂_T	Time derivative operator for mean quantities ($\partial_t = \partial_\tau + \alpha^m \partial_T$)
$\dot{\nabla}$	Lateral gradient for wave quantities
$\overline{\nabla}$	Lateral gradient for mean quantities ($\nabla = \dot{\nabla} + \alpha^n \overline{\nabla}$)

Table 3: Comparison of the scalings of the low-pass filtered equations where $\alpha \ll 1$ is the surface slope of waves.

	Section 3	Appendix C	Appendix C	Section 4
Equation system	Eqs. (23a-c)	Eqs. (C2a-c)	Eqs. (C4a-c)	Eqs. (C4a,c) & (49)
Coefficient of ∂_T	α^2	α^{n+2}	α^n	α^4
Coefficient of $\overline{\nabla}$	-	α^n	α^n	α^4
$\widehat{\mathbf{V}}$, $\overline{\mathbf{V}}$, $\overline{\mathbf{V}}^\varepsilon$	α^2	α^2	α^2	α^2
$\widehat{\varpi}$, $\widehat{w}$, $\overline{w}$, $\overline{w}^\varepsilon$	α^2	α^{n+2}	α^{n+2}	α^6
Horizontal momentum equation	α^4	α^{n+4}	α^{n+2}	α^6
Vertical momentum equation	α^4	α^4	α^2	α^2
$\nabla\overline{\eta}$	α^4	α^{n+4}	α^{n+2}	α^6
ϖ''	α^5	α^{n+5}	α^{n+3}	α^7
f/σ	-	-	-	α^4
$\nu\kappa^2/\sigma$	-	-	-	α^4

Table 4: The rule of numeric subscript in the present study, which represents summation for a given order of asymptotic expansion in terms of α.

$(AB)_2 = A_1B_1$
$(AB)_3 = A_1B_2 + A_2B_1$
$(AB)_4 = A_1B_3 + A_2B_2 + A_3B_1$
$(ABC)_4 = A_2B_1C_1 + A_1B_2C_1 + A_1B_1C_2$

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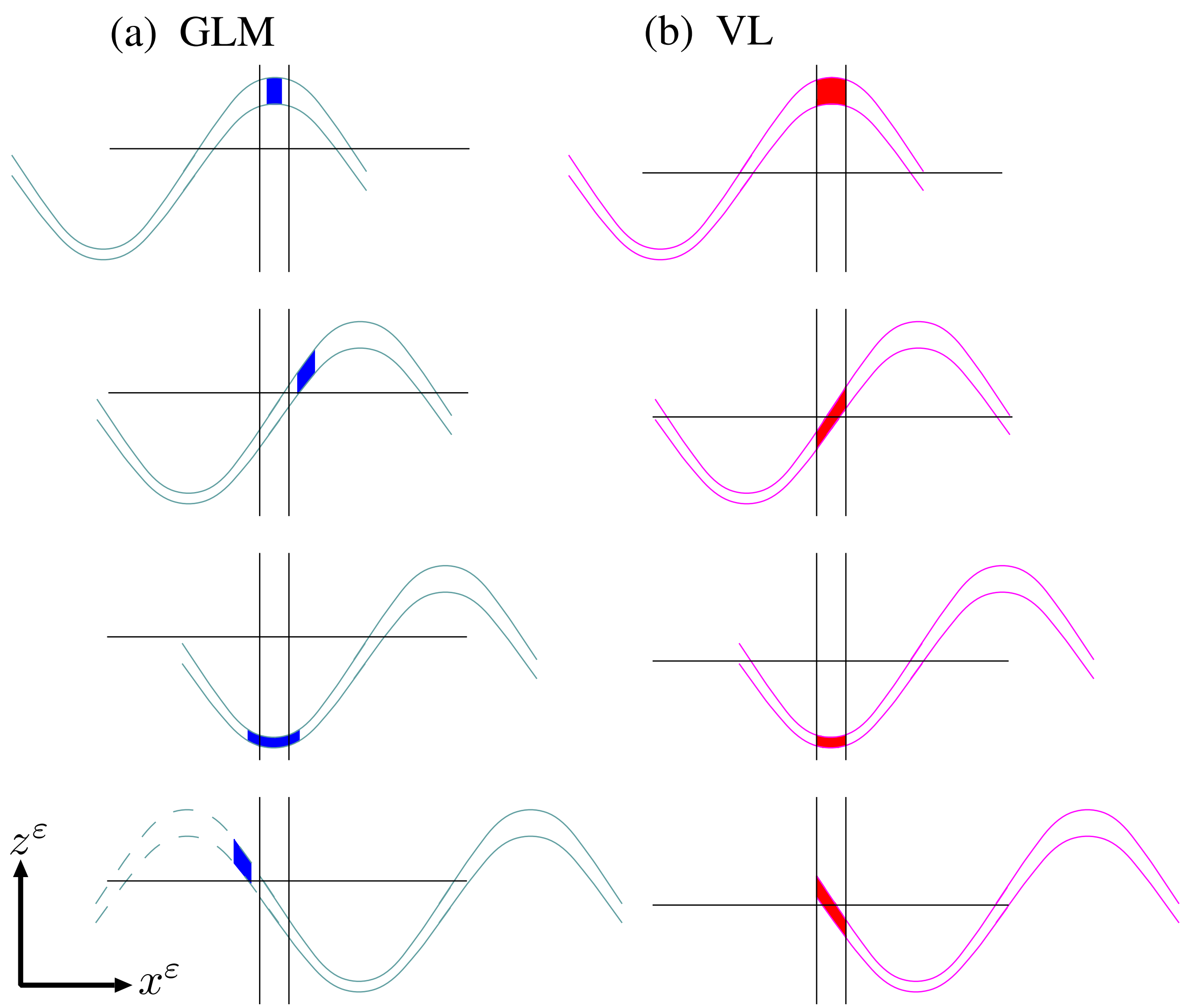

Figure 1: Illustration of the phase cycle of a wave propagating in the direction of the x^ε-axis. A control volume element in (a) the generalized-Lagrangian-mean (GLM) coordinates of AM78 and (b) the vertically Lagrangian (VL) coordinates of the present study is shaded in blue and red, respectively, with its low-pass filtered height, as measured in each coordinate system, being indicated by horizontal lines, and the reference horizontal position being indicated by vertical lines. Each color line indicates a material surface which is formed by connecting the instantaneous position of water particles whose three-dimensionally Lagrangian low-pass filtered height is a given value. Adapted from AG13.

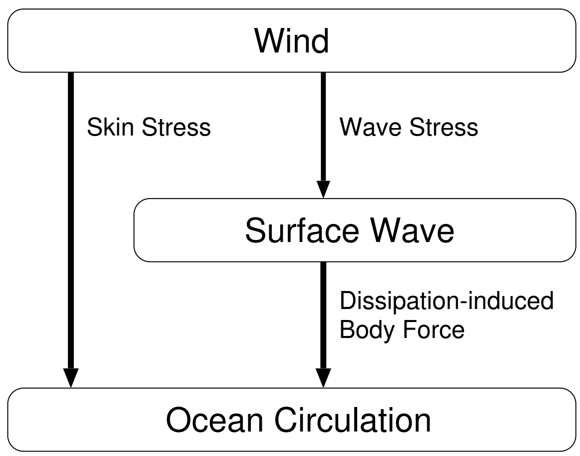

Figure 2: Schematic of momentum transfer between wind, surface waves, and ocean circulation.