Supplementary information for "A divergence-form wave-induced pressure inherent in the extension of the Eliassen-Palm theory to a three-dimensional framework for all waves at all latitudes" by H. Aiki, K. Takaya, and R. J. Greatbatch

D. Details of the derivation of some equations in the main manuscript

Equations (18a)-(18b) have been derived as follows

$$\underbrace{(-u'\xi_x' - v'\eta_x' + \zeta'\pi_{zx}')/2}_{\text{CE pseudomomentum}} = [u'(\eta_y' + \zeta_z') - v'\eta_x' - \zeta_z'\pi_x' + (\zeta'\pi_x')_z]/2$$

$$= [u'\eta_y' + 2\zeta_z'u' - v'\eta_x' - \zeta_z'f\eta' + (\zeta'\pi_x')_z]/2$$

$$= \underbrace{\zeta_z'u' + q'\eta'/2}_{\text{IB pseudomomentum}} + [(u'\eta')_y - (v'\eta')_x + (\zeta'\pi_x')_z]/2, \tag{18a}$$

$$\underbrace{(-u'\xi_y' - v'\eta_y' + \zeta'\pi_{zy}')/2}_{\text{CE pseudomomentum}} = [-u'\xi_y' + v'(\xi_x' + \zeta_z') + \zeta_z'\pi_y' + (\zeta'\pi_y')_z]/2$$

$$= [-u'\xi_y' + v'\xi_x' + 2\zeta_z'v' + \zeta_z'f\xi' + (\zeta'\pi_y')_z]/2$$

$$= \underbrace{\zeta_z'v' - q'\xi'/2}_{\text{IB pseudomomentum}} + [(v'\xi')_x - (u'\xi')_y + (\zeta'\pi_y')_z]/2. \tag{18b}$$

Equation (23a) has been derived by multiplying (6a), (6b), (21c), (21a), (21b), $\zeta_t' = w'$ by $-\xi_x'/2$, $-\eta_x'/2$, $\zeta'/2$, $-u'/2$, $-v'/2$, $\pi_{zx}'/2$, respectively, and then taking the sum of the six equations to yield

$$\left[\underbrace{(-u'\xi_x' - v'\eta_x' + \zeta'\pi_{zx}')/2}_{\text{CE pseudomomentum}}\right]_t$$

$$= (\xi_x'p_x' + \eta_x'p_y' - \zeta'N^2\zeta_x')/2 + (u'\pi_{xx}' + v'\pi_{yx}' + w'\pi_{zx}')/2$$

$$= -\nabla \cdot \underbrace{\langle\!\langle -(\xi_x'p' + u'\pi_x')/2, -(\eta_x'p' + v'\pi_x')/2, -(\zeta_x'p' + w'\pi_x')/2 \rangle\!\rangle}_{\text{XCE flux}}. \tag{23a}$$

Equation (23b) has been derived by multiplying (6a), (6b), (22c), (22a), (22b), $\zeta_t' = w'$ by $-\xi_y'/2$,

$-\eta'_y/2$, $\zeta'/2$, $-u'/2$, $-v'/2$, $\pi'_{zy}/2$, respectively, and then taking the sum of the six equations to yield,

$$\underbrace{[(-u'\xi'_y - v'\eta'_y + \zeta'\pi'_{zy})/2]}_{\text{CE pseudomomentum}}{}_t$$

$$= (\xi'_y p'_x + \eta'_y p'_y - \zeta' N^2 \zeta'_y)/2 + (u'\pi'_{xy} + v'\pi'_{yy} + w'\pi'_{zy})/2 + \beta(v'\xi' - u'\eta')/2$$

$$= -\nabla \cdot \underbrace{\langle\!\langle -(\xi'_y p' + u'\pi'_y)/2, -(\eta'_y p' + v'\pi'_y)/2, -(\zeta'_y p' + w'\pi'_y)/2\rangle\!\rangle}_{\text{YCE flux}} + \beta(v'\xi' - u'\eta')/2. \qquad (23\text{b})$$

Equations (27a) has been derived as follows.

$$(\underbrace{\zeta'_z u' + q'\eta'/2}_{\text{IB pseudomomentum}})_t = \zeta'_{zt}u' + \zeta'_z(-p'_x + fv') + \underbrace{q'_t}_{-\beta v'}\eta'/2 + \underbrace{q'}_{-\beta\eta'}v'/2$$

$$= w'_z u' - \zeta'_z p'_x + \underbrace{(f\zeta'_z - \beta\eta')}_{v'_x - u'_y}v'$$

$$= -(u'_x + v'_y)u' - \zeta'_z p'_x + (v'_x - u'_y)v'$$

$$= -(u'u')_x - (v'u')_y + K_x - \zeta'_z p'_x$$

$$= -(u'u')_x - (v'u')_y + K_x + \zeta' p'_{xz} - (\zeta' p'_x)_z$$

$$= -\nabla \cdot \underbrace{\langle\!\langle \overbrace{E - v'v'}^{u'u'-K+G}, v'u', \zeta' p'_x \rangle\!\rangle}_{\text{XIB flux}}, \qquad (27\text{a})$$

where $K \equiv (u'^2 + v'^2)/2$ is the wave kinetic energy and $G \equiv (N^2/2)\zeta'^2$ is the wave potential energy. The second line of (27a) has been derived using (10)-(11), and the last line has been derived using both (9) and $E = K + G$.

Equation (27b) has been derived as follows,

$$(\underbrace{\zeta'_z v' - q'\xi'/2}_{\text{IB pseudomomentum}})_t = \zeta'_{zt} v' + \zeta'_z(-p'_y - fu') - \underbrace{q'_t}_{-\beta v'}\xi'/2 - \underbrace{q'}_{-\beta\eta'}u'/2$$

$$= w'_z v' - \zeta'_z p'_y - \underbrace{(f\zeta'_z - \beta\eta')}_{v'_x - u'_y}u' + \beta(v'\xi' - u'\eta')/2$$

$$= -(u'_x + v'_y)v' - \zeta'_z p'_y - (v'_x - u'_y)u' + \beta(v'\xi' - u'\eta')/2$$

$$= -(u'v')_x - (v'v')_y + K_y - \zeta'_z p'_y + \beta(v'\xi' - u'\eta')/2$$

$$= -(u'v')_x - (v'v')_y + K_y + \zeta' p'_{yz} - (\zeta' p'_y)_z + \beta(v'\xi' - u'\eta')/2$$

$$= -\nabla \cdot \underbrace{\langle\!\langle v'u', \overbrace{E - u'u'}^{v'v' - K + G}, \zeta' p'_y \rangle\!\rangle}_{\text{YIB flux}} + \beta(v'\xi' - u'\eta')/2, \tag{27b}$$

where the second line has been derived using (10)-(11), and the last line has been derived using both (9) and $E = K + G$.

Equations (33a)-(33f) have been derived as follows.

$$(\xi'p'_x - u'\pi'_x)/2 - \Lambda = [u'(u' - f\eta') - \xi'(u'_t - fv')]/2 - \Lambda$$

$$= E - (v'v' - \eta'v'_t)/2$$

$$= E - v'v' + (v'\eta')_t/2, \tag{33a}$$

$$(\eta'p'_x - v'\pi'_x)/2 = [v'(f\eta' - \pi'_x) - \eta'(fv' - p'_x)]/2$$

$$= (v'u' - \eta'u'_t)/2$$

$$= v'u' - (u'\eta')_t/2, \tag{33b}$$

$$(\zeta'p'_x - w'\pi'_x)/2 = \zeta'p'_x - (\zeta'\pi'_x)_t/2, \tag{33c}$$

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$$(\xi' p'_y - u' \pi'_y)/2 = [u'(-f\xi' - \pi'_y) - \xi'(-fu' - p'_y)]/2$$

$$= (u'v' - \xi' v'_t)/2$$

$$= u'v' - (v'\xi')_t/2, \tag{33d}$$

$$(\eta' p'_y - v' \pi'_y)/2 - \Lambda = [v'(v' + f\xi') - \eta'(v'_t + fu')]/2 - \Lambda$$

$$= E - (u'u' - \xi' u'_t)/2$$

$$= E - u'u' + (u'\xi')_t/2, \tag{33e}$$

$$(\zeta' p'_y - w' \pi'_y)/2 = \zeta' p'_y - (\zeta' \pi'_y)_t/2, \tag{33f}$$

where the first line of each of (33a) and (33e) has been derived using the set of (6a)-(6b) and (16a)-(16b), and the second line of each of (33a) and (33e) has been derived using (32).

Equations (49a)-(49b) have been derived as follows,

$$\underbrace{\overline{\zeta' u'_z} - \overline{\eta' q'}/2}_{\text{DI wave activity}} = \overline{\zeta' u'_z} + \overline{\eta' q'}/2 - \overline{\eta' q'}$$

$$= \overline{\zeta' u'_z} - \beta \overline{\eta'^2}/2 - \overline{\eta'(v'_x - u'_y - f\zeta'_z)}$$

$$= -(\overline{\xi' u' + \eta' v'})_x - (f\overline{\xi'\eta'})_x/2 - (f\overline{\eta'^2})_y/2$$

$$+ \underbrace{(\overline{\xi' u'_x + \eta' u'_y + \zeta' u'_z})}_{u^{Stokes}} - \underbrace{[-u'\xi'_x - v'\eta'_x + f(\xi'_x \eta' - \xi' \eta'_x)/2]}_{\text{GL pseudomomentum}}, \tag{49a}$$

$$\underbrace{\overline{\zeta' v'_z} + \overline{\xi' q'}/2}_{\text{DI wave activity}} = \overline{\zeta' v'_z} - \overline{\xi' q'}/2 + \overline{\xi' q'}$$

$$= \overline{\zeta' v'_z} + \beta \overline{\xi'\eta'}/2 + \overline{\xi'(v'_x - u'_y - f\zeta'_z)}$$

$$= -(\overline{\xi' u' + \eta' v'})_y + (f\overline{\xi'^2})_x/2 + (f\overline{\xi'\eta'})_y/2$$

$$+ \underbrace{(\overline{\xi' v'_x + \eta' v'_y + \zeta' v'_z})}_{v^{Stokes}} - \underbrace{[-u'\xi'_y - v'\eta'_y + f(\xi'_y \eta' - \xi' \eta'_y)/2]}_{\text{GL pseudomomentum}}, \tag{49b}$$

where the second line of each equation has been derived using both (11) and $q' \equiv v'_x - u'_y - f\zeta'_z$, and u^{Stokes} and v^{Stokes} on the last line have been defined in (12a)-(12b).

E. Nonhydrostatic IB pseudomomentum and wave-activity

This section investigates the first part of the DI wave-activity in (20a)-(20b). Below we show that $\langle\!\langle \overline{\zeta'u'_z}, \overline{\zeta'v'_z}\rangle\!\rangle$ represents a hydrostatic approximation for $\langle\!\langle \overline{\zeta'(u'_z - w'_x)}, \overline{\zeta'(v'_z - w'_y)}\rangle\!\rangle$ that has been used in the nonhydrostatic gravity wave literature, in the limit of no vertical shear of mean flows. See Eq. (6.16) of SS92 for the expression in the presence of the vertical shear of mean flows.

An equation system for nonhydrostatic linear waves in a rotating stratified fluid may be written as,

$$u'_t - fv' = -(p' + p^n)_x, \tag{E1a}$$

$$v'_t + fu' = -(p' + p^n)_y, \tag{E1b}$$

$$w'_t = -p^n_z, \tag{E1c}$$

where p' is hydrostatic pressure, which is defined by (6d), and p^n is nonhydrostatic pressure. The Coriolis parameter is $f = f_0 + \beta y$. Using (8) and nonhydrostatic versions of (16a)-(16b), we expand the zonal and meridional components of the nonhydrostatic version of the CE pseudomomentum in (17) to read,

$$\underbrace{(-u'\xi'_x - v'\eta'_x - w'\zeta'_x + \zeta'\pi'_{zx})/2}_{\text{nonhydrostatic CE pseudomomentum}}$$

$$= [u'(\eta'_y + \zeta'_z) - v'\eta'_x - w'\zeta'_x - \zeta'_z\pi'_x + (\zeta'\pi'_x)_z]/2$$

$$= [u'\eta'_y + 2\zeta'_z u' - v'\eta'_x - w'\zeta'_x + \zeta'_z(-f\eta' + \pi^n_x) + (\zeta'\pi'_x)_z]/2$$

$$= \underbrace{\zeta'_z u' + \zeta'w'_x + q'\eta'/2}_{\text{nonhydrostatic IB pseudomomentum}} + [(u'\eta')_y - (v'\eta' + w'\zeta')_x + (\zeta'\pi'_x + \zeta'\pi^n_x)_z]/2, \tag{E2a}$$

$$\underbrace{(-u'\xi_y' - v'\eta_y' - w'\zeta_y' + \zeta'\pi_{zy}')/2}_{\text{nonhydrostatic CE pseudomomentum}}$$

$$= [-u'\xi_y' + v'(\xi_x' + \zeta_z') - w'\zeta_y' - \zeta_z'\pi_y' + (\zeta'\pi_y')_z]/2$$

$$= [-u'\xi_y' + v'\xi_x' + 2\zeta_z'v' - w'\zeta_y' + \zeta_z'(f\xi' + \pi_y^n) + (\zeta'\pi_y')_z]/2$$

$$= \underbrace{\zeta_z'v' + \zeta'w_y' - q'\xi'/2}_{\text{nonhydrostatic IB pseudomomentum}} + [(v'\xi')_x - (u'\xi' + w'\zeta')_y + (\zeta'\pi_y' + \zeta'\pi_y^n)_z]/2, \qquad \text{(E2b)}$$

where (8) and the nonhydrostatic version of (16a)-(16b) have been used with $\pi^n \equiv \int^t p^n dt$. Equations (E2a)-(E2b) correspond to (18a)-(18b). As in (20a)-(20b), we suggest to refer the difference of the quasi-Stokes velocity and the nonhydrostatic IB pseudomomentum as the nonhydrostatic DI wave-activity,

$$\underbrace{(\overline{\zeta'u'})_z}_{u^{qs}} - \underbrace{(\overline{\zeta_z'u' + \zeta'w_x'} + \overline{\eta'q'}/2)}_{\text{nonhydrostatic IB pseudomomentum}} = \underbrace{\overline{\zeta'(u_z' - w_x')} - \overline{\eta'q'}/2}_{\text{nonhydrostatic DI wave-activity}}, \qquad \text{(E3a)}$$

$$\underbrace{(\overline{\zeta'v'})_z}_{v^{qs}} - \underbrace{(\overline{\zeta_z'v' + \zeta'w_y'} - \overline{\xi'q'}/2)}_{\text{nonhydrostatic IB pseudomomentum}} = \underbrace{\overline{\zeta'(v_z' - w_y')} + \overline{\xi'q'}/2}_{\text{nonhydrostatic DI wave-activity}}, \qquad \text{(E3b)}$$

where the first term on the right hand side is indeed the same as the wave-activity used in the gravity wave literature except that the vertical shear of mean flows is assumed to be absent in the present study. We conclude that $\langle\langle \overline{\zeta'u_z'}, \overline{\zeta'v_z'} \rangle\rangle$, which is the first part of the DI wave-activity in (20a)-(20b), represents a hydrostatic approximation for $\langle\langle \overline{\zeta'(u_z' - w_x')}, \overline{\zeta'(v_z' - w_y')} \rangle\rangle$.

For readers who might be interested, we show prognostic equations for the nonhydrostatic IB pseudomomentum to read,

$$[\underbrace{\zeta_z'u' + \zeta'w_x' + q'\eta'/2}_{\text{nonhydrostatic IB pseudomomentum}}]_t$$
$$= -[u'u' - (u'^2 + v'^2 + w'^2 - N^2\zeta'^2)/2]_x - [v'u']_y - [\zeta'(p' + p^n)_x]_z, \qquad \text{(E4a)}$$

$$[\underbrace{\zeta_z'v' + \zeta'w_y' - q'\xi'/2}_{\text{nonhydrostatic IB pseudomomentum}}]_t$$
$$= -[v'u']_x - [v'v' - (u'^2 + v'^2 + w'^2 - N^2\zeta'^2)/2]_y - [\zeta'(p' + p^n)_y]_z + \beta(v'\xi' - u'\eta')/2. \text{ (E4b)}$$

Equations (E4a)-(E4b) have been derived using (10)-(11) and (E1a)-(E1c). We have omitted the

details of the derivation of (E4b)-(E4b), because it is essentially the same as that in the hydrostatic IB pseudomomentum equations (27a)-(27b).