A Van Benthem Theorem for Modal Team Semantics*

Juha Kontinen[1], Julian-Steffen Müller[2], Henning Schnoor[3], and Heribert Vollmer[2]

1   University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, 00014 Helsinki, Finland
2   Leibniz Universität Hannover, Institut für Theoretische Informatik, Appelstr. 4, 30167 Hannover, Germany
3   Institut für Informatik, Christian-Albrechts-Universität zu Kiel, 24098 Kiel, Germany

Abstract

The famous van Benthem theorem states that modal logic corresponds exactly to the fragment of first-order logic that is invariant under bisimulation. In this article we prove an exact analogue of this theorem in the framework of modal dependence logic MDL and team semantics. We show that modal team logic MTL, extending MDL by classical negation, captures exactly the FO-definable bisimulation invariant properties of Kripke structures and teams. We also compare the expressive power of MTL to most of the variants and extensions of MDL recently studied in the area.

Keywords and phrases   modal logic, dependence logic, team semantics, expressivity, bisimulation, independence, inclusion, generalized dependence atom

1   Introduction

The concepts of dependence and independence are ubiquitous in many scientific disciplines such as experimental physics, social choice theory, computer science, and cryptography. Dependence logic D [22] and its so-called team semantics have given rise to a new logical framework in which various notions of dependence and independence can be formalized and studied. Dependence logic extends first-order logic by dependence atoms

$$=(x_1, \ldots, x_{n-1}, x_n), \tag{1}$$

expressing that the value of the variable x_n is functionally dependent on the values of $x_1, \ldots, x_{n-1}$. The formulas of dependence logic are evaluated over *teams*, i.e., sets of assignments, and not over single assignments as in first-order logic.

In [23] a modal variant of dependence logic MDL was introduced. In the modal framework teams are sets of worlds, and a dependence atom

$$=(p_1, \ldots, p_{n-1}, p_n) \tag{2}$$

holds in a team T if there is a Boolean function that determines the value of the propositional variable p_n from those of $p_1, \ldots, p_{n-1}$ in all worlds in T. One of the fundamental properties of MDL (and of dependence logic) is that its formulas satisfy the so-called downwards closure property: if $M, T \models \varphi$, and $T' \subseteq T$, then $M, T' \models \varphi$. Still, the modal framework is very

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different from the first-order one, e.g., dependence atoms between propositional variables can be eliminated with the help of the classical disjunction $\veebar$ [23]. On the other hand, it was recently shown that eliminating dependence atoms using disjunction causes an exponential blow-up in the formula size, that is, any formula of $\mathsf{ML}(\veebar)$ logically equivalent to the atom in (2) is bound to have length exponential in n [11]. The central complexity theoretic questions regarding MDL have been solved in [21, 15, 3, 16].

Extended modal dependence logic, EMDL, was introduced in [4]. This extension is defined simply by allowing ML formulas to appear inside dependence atoms, instead of only propositions. EMDL can be seen as the first step towards combining dependencies with temporal reasoning. EMDL is strictly more expressive than MDL but its formulas still have the downwards closure property. In fact, EMDL has recently been shown to be equivalent to the logic $\mathsf{ML}(\veebar)$ [4, 11].

In the first-order case, several interesting variants of the dependence atoms have been introduced and studied. The focus has been on independence atoms

$$(x_1, \ldots, x_\ell) \perp_{(y_1, \ldots, y_m)} (z_1, \ldots, z_n),$$

and inclusion atoms

$$(x_1, \ldots, x_\ell) \subseteq (y_1, \ldots, y_\ell),$$

which were introduced in [9] and [5], respectively. The intuitive meaning of the independence atom is that the variables $x_1, \ldots, x_\ell$ and $z_1, \ldots, z_n$ are independent of each other for any fixed value of $y_1, \ldots, y_m$, whereas the inclusion atom declares that all values of the tuple $(x_1, \ldots, x_\ell)$ appear also as values of $(y_1, \ldots, y_\ell)$. In [12] a modal variant, MIL, of independence logic was introduced. The logic MIL contains MDL as a proper sublogic, in particular, its formulas do not in general have the downwards closure property. In [12] it was also noted that all MIL formulas are invariant under bisimulation when this notion is lifted from single worlds to a relation between sets of words in a natural way. At the same time (independently) in [11] it was shown that EMDL and $\mathsf{ML}(\veebar)$ can express exactly those properties of Kripke structures and teams that are downwards closed and invariant under k-bisimulation for some $k \in \mathbb{N}$.

A famous theorem by Johan van Benthem [24, 25] states that modal logic is exactly the fragment of first-order logic that is invariant under (full) bisimulation. In this paper we study the analogues of this theorem in the context of team semantics. Our main result shows that an analogue of the van Benthem theorem for team semantics can be obtained by replacing ML by *Modal Team Logic* (MTL). MTL was introduced in [17] and extends ML (and MDL) by classical negation $\sim$. More precisely, we show that for any team property P the following are equivalent:

(i) There is an MTL-formula which expresses P,
(ii) there is a first-order formula which expresses P and P is bisimulation-invariant,
(iii) P is invariant under k-bisimulation for some k,
(iv) P is bisimulation-invariant and local.

We also study whether all bisimulation invariant properties can be captured by natural variants of EMDL. We consider extended modal independence and extended modal inclusion logic (EMIL and EMINCL, respectively), which are obtained from EMDL by replacing the dependence atom with the independence (resp. inclusion) atom. We show that both of these logics fail to capture all bisimulation invariant properties, and therefore in particular are strictly weaker than MTL. On the other hand, we show that $\mathsf{EMINCL}(\veebar)$ (EMINCL extended with classical disjunction) is in fact as expressive as MTL, but the analogously

defined $\mathsf{EMIL}(\lozenge)$ is strictly weaker. Finally, we show that the extension $\mathsf{ML}^{\mathsf{FO}}$ of ML by all first-order definable generalized dependence atoms (see [12]) gives rise to a logic that is as well equivalent to MTL.

2 Preliminaries

A *Kripke model* is a tuple $M = (W, R, \pi)$ where W is a nonempty set of worlds, $R \subseteq W \times W$, and $\pi \colon P \to 2^W$, where P is the set of propositional variables. A *team* of a model M as above is simply a set $T \subseteq W$. The central basic concept underlying Väänänen's modal dependence logic and all its variants is that modal formulas are evaluated not in a world but in a team. This is made precise in the following definitions. We first recall the usual syntax of modal logic ML:

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \lozenge \varphi \mid \square \varphi,$$

where p is a propositional variable. Note that we consider only formulas in negation normal form, i.e., negation appears only in front of atoms. As will become clear from the definition of team semantics of ML, that we present next, p and $\neg p$ are not dual formulas, consequently *tertium non datur* does not hold in the sense that it is possible that $M, T \not\models p$ and $M, T \not\models \neg p$ (however, we still have that $M, T \models p \vee \neg p$ for all models M and teams T). It is worth noting that in [23], the connective $\neg$ is allowed to appear freely in MDL formulas (with semantics generalizing the atomic negation case of Definition 2.1 below, note that classical negation as allowed in MTL is not allowed in MDL). The well-known dualities from classical modal logic are also true for MDL formulas hence any ML-formula (even MDL) can be rewritten in such a way that $\neg$ only appears in front of propositional variables.

▶ **Definition 2.1.** Let $M = (W, R, \pi)$ be a Kripke model, let $T \subseteq W$ be a team, and let φ be an ML-formula. We define when $M, T \models \varphi$ holds inductively:

- If $\varphi = p$, then $M, T \models \varphi$ iff $T \subseteq \pi(p)$,
- If $\varphi = \neg p$, then $M, T \models \varphi$ iff $T \cap \pi(p) = \emptyset$,
- If $\varphi = \psi \vee \chi$ for some formulas ψ and χ, then $M, T \models \varphi$ iff $T = T_1 \cup T_2$ with $M, T_1 \models \psi$ and $M, T_2 \models \chi$,
- If $\varphi = \psi \wedge \chi$ for some formulas ψ and χ, then $M, T \models \varphi$ iff $M, T \models \psi$ and $M, T \models \chi$,
- If $\varphi = \lozenge \psi$ for some formula ψ, then $M, T \models \varphi$ iff there is some team T' of M such that $M, T' \models \psi$ and **1.** for each $w \in T$, there is some $w' \in T'$ with $(w, w') \in R$, and **2.** for each $w' \in T'$, there is some $w \in T$ with $(w, w') \in R$.
- If $\varphi = \square \psi$ for some formula ψ, then $M, T \models \varphi$ iff $M, T' \models \psi$, where T' is the set $\{w' \in M \mid (w, w') \in R \text{ for some } w \in T\}$.

Analogously to the first-order setting, ML-formulas satisfy the following *flatness* property [22]. Here, the notation $M, w \models \varphi$ in item 3 refers to the standard semantics of modal logic (without teams).

▶ **Proposition 2.2.** *Let M be a Kripke model and T a team of M. Let φ be an ML-formula. Then the following are equivalent:*

1. *$M, T \models \varphi$,*
2. *$M, \{w\} \models \varphi$ for each $w \in T$,*
3. *$M, w \models \varphi$ for each $w \in T$.*

Modal team logic extends ML by a second type of negation, denoted by $\sim$, and interpreted just as classical negation. The syntax is formally given as follows:

$$\varphi ::= p \mid \neg p \mid \sim\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where p is a propositional variable. The semantics of MTL is defined by extending Def. 2.1 by the following clause:

- If $\varphi = \sim\psi$ for some formula ψ, then $M, T \models \varphi$ iff $M, T \not\models \psi$.

We note that usually (see [17]), MTL also contains dependence atoms; however since these atoms can be expressed in MTL we omit them in the syntax (see Proposition 2.3 below). The classical disjunction $\oslash$ (in some other context also referred to as *intuitionistic disjunction*) is also readily expressed in MTL: $\varphi \oslash \psi$ is logically equivalent to $\sim(\sim\varphi \wedge \sim\psi)$.

For an ML formula φ, we let φ^{dual} denote the formula that is obtained by transforming $\neg\varphi$ to negation normal form. Now by Proposition 2.2 it follows that

$$M, T \models \varphi^{dual} \text{ iff } M, w \not\models \varphi \text{ for all } w \in T,$$

hence $M, T \models \sim\psi^{dual}$ if and only if there is some $w \in T$ with $M, w \models \psi$. We therefore often write $\mathsf{E}\psi$ instead of $\sim\psi^{dual}$. Note that E is not a global operator stating existence of a world anywhere in the model, but E is evaluated in the current team. It is easy to see (and follows from Proposition 2.8) that a global "exists" operator cannot be expressed in MTL.

The next proposition shows that dependence atoms can be easily expressed in MTL.

▶ **Proposition 2.3.** *The dependence atom* (2) *can be expressed in* MTL *by a formula that has length polynomial in n.*

Proof. Note first that, analogously to the first-order case [1], (2) is logically equivalent with

$$\left(\bigwedge_{1 \le i \le n-1} =(p_i) \right) \to \, =(p_n) \,,$$

where $\to$ is the so-called intuitionistic implication with the following semantics:

$$M, T \models \varphi \to \psi \text{ iff for all } T' \subseteq T: \text{ if } M, T' \models \varphi \text{ then } M, T' \models \psi.$$

The connective $\to$ has a short logically equivalent definition in MTL (see [17]): $\varphi \to \psi$ is equivalent to $(\sim\varphi \oslash \psi) \otimes \bot$, where $\otimes$ is the dual of $\vee$, i.e., $\varphi \otimes \psi := \sim(\sim\varphi \vee \sim\psi)$, and $\bot$ is a shorthand for the formula $p_0 \wedge \neg p_0$. Finally, $=(p_i)$ can be written as $p_i \oslash \neg p_i$, hence the claim follows.   ◀

The intuitionistic implication used in the proof above has been studied in the modal team semantics context in [26].

We now introduce the central concept of bisimulation [19, 25]. Intuitively, two *pointed models* (i.e., pairs of models and worlds from the model) (M_1, w_1) and (M_2, w_2) are bisimilar, if they are indistinguishable from the point of view of modal logic. The notion of k-bisimilarity introduced below corresponds to indistinguishability by formulas with modal depth up to k: For a formula φ in any of the logics considered in this paper, the *modal depth* of φ, denoted with $md(\varphi)$, is the maximal nesting degree of modal operators (i.e., $\Box$ and $\Diamond$) in φ.

▶ **Definition 2.4.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models. We define inductively what it means for states $w_1 \in W_1$ and $w_2 \in W_2$ to be k-bisimilar, for some $k \in \mathbb{N}$, written as $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$.

- $(M_1, w_1) \rightleftharpoons_0 (M_2, w_2)$ holds if for each propositional variable p, we have that $M_1, w_1 \models p$ if and only if $M_2, w_2 \models p$.
- $(M_1, w_1) \rightleftharpoons_{k+1} (M_2, w_2)$ holds if the following three conditions are satisfied:
    1. $(M_1, w_1) \rightleftharpoons_0 (M_2, w_2)$,
    2. for each successor w_1' of w_1 in M_1, there is a successor w_2' of w_2 in M_2 such that $(M_1, w_1') \rightleftharpoons_k (M_2, w_2')$ (*forward* condition),
    3. for each successor w_2' of w_2 in M_2, there is a successor w_1' of w_1 in M_1 such that $(M_1, w_1') \rightleftharpoons_k (M_2, w_2')$ (*backward* condition).

Full bisimilarity is defined similarly: Pointed models (M_1, w_1) and (M_2, w_2) are *bisimilar* if there is a relation $Z \subseteq W_1 \times W_2$ such that $(w_1, w_2) \in Z$, and for all $(w_1, w_2) \in Z$, we have that w_1 and w_2 satisfy the same propositional variables, and for each successor w_1' of w_1 in M_1, there is a successor w_2' of w_2 in M_2 with $(w_1', w_2') \in Z$ (forward condition), and analogously for each successor w_2' of w_2 in M_2, there is a successor w_1' of w_1 in M_1 with $(w_1', w_2') \in Z$ (back condition). In this case we simply say that M_1, w_1 and M_2, w_2 are *bisimilar*. It is easy to see that bisimilarity implies k-bisimilarity for each k.

▶ **Definition 2.5.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, and let $w_1 \in W_1$, $w_2 \in W_2$. Then (M_1, w_1) and (M_2, w_2) are k-equivalent for some $k \in \mathbb{N}$, written $(M_1, w_1) \equiv_k (M_2, w_2)$ if for each modal formula φ with $md(\varphi) \le k$, we have that $M_1, w_1 \models \varphi$ if and only if $M_2, w_2 \models \varphi$.

Again, we simply write $w_1 \equiv_k w_2$ if the models M_1 and M_2 are clear from the context. As mentioned above, k-bisimilarity and k-equivalence coincide. The following result is standard (see, e.g., [2]):

▶ **Proposition 2.6.** *Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, and let $w_1 \in W_1$, $w_2 \in W_2$. Then $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$ if and only if $(M_1, w_1) \equiv_k (M_2, w_2)$.*

For MTL and more generally logics with team semantics, the above notion of bisimulation can be lifted to teams. The following definition is a natural adaptation of k-bisimilarity to the team setting:

▶ **Definition 2.7.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, let T_1 and T_2 be teams of M_1 and M_2. Then (M_1, T_1) and (M_2, T_2) are k-bisimilar, written as $M_1, T_1 \rightleftharpoons_k M_2, T_2$ if the following holds:

- for each $w_1 \in T_1$, there is some $w_2 \in T_2$ such that $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$,
- for each $w_2 \in T_2$, there is some $w_1 \in T_1$ such that $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$.

Full bisimilarity on the team level is defined analogously. In this case we again say that (M_1, T_1) and (M_2, T_2) are bisimilar, and write $M_1, T_1 \rightleftharpoons M_2, T_2$, if there is a relation $Z \subseteq W_1 \times W_2$ satisfying the forward and backward conditions as above, and which additionally satisfies that for each $w_1 \in T_1$, there is some $w_2 \in T_2$ with $(w_1, w_2) \in Z$, and for each $w_2 \in T_2$, there is some $w_1 \in T_1$ with $(w_1, w_2) \in Z$. This notion of team-bisimilarity was first introduced in [12] and [11]. The following result is easily proved by induction on the formula length:

▶ **Proposition 2.8.** *Let M_1 and M_2 be Kripke models, let T_1 and T_2 be teams of M_1 and of M_2. Then*

1. *If $(M_1, T_1) \rightleftharpoons_k (M_2, T_2)$, then for each formula $\varphi \in$ MTL with $md(\varphi) \le k$, we have that $M_1, T_1 \models \varphi$ if and only if $M_2, T_2 \models \varphi$.*

2. *If $(M_1, T_1) \rightleftharpoons (M_2, T_2)$, then for each formula $\varphi \in \mathsf{MTL}$, we have that $M_1, T_1 \models \varphi$ if and only if $M_2, T_2 \models \varphi$.*

Proof. It suffices to show the first claim, since $(M_1, T_1) \rightleftharpoons (M_2, T_2)$ implies $(M_1, T_1) \rightleftharpoons_k (M_2, T_2)$ for every $k \in \mathbb{N}$. We show the first claim by induction over the construction of φ. In this proof, we omit the models in the notation $\rightleftharpoons_k$ (for the entire proof, the model on the left-hand side of $\rightleftharpoons_k$ is always M_1, and the model on the right-hand side is always M_2). Clearly, it suffices to show that if $M_1, T_1 \models \varphi$, then $M_2, T_2 \models \varphi$. Hence assume that $M_1, T_1 \models \varphi$

- Assume that $\varphi = p$ for a variable p (the case $\varphi = \neg p$ is analogous). Since $M_1, T_1 \models \varphi$, we know that $M_1, w_1 \models p$ for each $w_1 \in T_1$. Let $w_2 \in T_2$. Since $T_1 \rightleftharpoons_k T_2$, there is some $w_1 \in T_1$ with $w_1 \rightleftharpoons_k w_2$. By the above, we know that $M_1, w_1 \models p$, and by definition of $\rightleftharpoons_k$, it follows that $M_2, w_2 \models p$. Hence $M_2, T_2 \models \varphi$ as required.
- Assume that $\varphi = \psi \wedge \chi$. Since $M_1 \models \varphi$, we know that $M_1, T_1 \models \psi$ and $M_1, T_1 \models \chi$. By induction, it follows that $M_2, T_2 \models \psi$ and $M_2, T_2 \models \chi$. Therefore, $M_2, T_2 \models \varphi$ as required.
- Assume that $\varphi = \psi \vee \chi$. Then $T_1 = T_1' \cup T_1''$ with $M_1, T_1' \models \psi$ and $M_1, T_1'' \models \chi$.
  Let $T_2' = \{w_2 \in T_2 \mid \text{there is some } w_1 \in T_1' \text{ with } w_1 \rightleftharpoons_k w_2\}$ and $T_2'' = \{w_2 \in T_2 \mid \text{there is some } w_1 \in T_1'' \text{ with } w_1 \rightleftharpoons_k w_2\}$. It can easily be verified that $T_2' \cup T_2'' = T_2$ and $T_1' \rightleftharpoons_k T_2'$ and $T_1'' \rightleftharpoons_k T_2''$. By induction, it follows that $M_2, T_2' \models \psi$ and $M_2, T_2'' \models \chi$, and hence $M_2, T_2 \models \varphi$ as required.
- Assume that $\varphi = {\sim}\psi$. Then $M_1, T_1 \not\models \psi$. Assume indirectly that $M_2, T_2 \models \psi$. Due to induction, it then follows that $M_1, T_1 \models \psi$, a contradiction.
- Assume that $\varphi = \Box\psi$, let $T_1' = \{w_1' \mid (w_1, w_1') \in R_1 \text{ for some } w_1 \in T_1\}$, and $T_2' = \{w_2' \mid (w_2, w_2') \in R_1 \text{ for some } w_2 \in T_2\}$. From $M_1, T_1 \models \varphi$, we know that $M_1, T_1' \models \psi$, and we need to show that $M_2, T_2' \models \psi$. Due to induction, and since $md(\psi) = md(\varphi) - 1 \leq k - 1$, it suffices to show that $T_1' \rightleftharpoons_{k-1} T_2'$.
  Hence let $w_1' \in T_1'$. Then w_1' is the R_1-successor of some $w_1 \in T_1$. Since $T_1 \rightleftharpoons_k T_2$, there is some $w_2 \in T_2$ with $w_1 \rightleftharpoons_k w_2$. In particular, there is some w_2' such that w_2' is an R_2-successor of w_2, and $w_1' \rightleftharpoons_k w_2'$. The other direction is symmetric.
- Assume that $\varphi = \Diamond\psi$. Since $M_1, T_1 \models \varphi$, there is a team T_1' of M_1 such that for each $w_1 \in T_1$, there is some $w_1' \in T_1'$ and $(w_1, w_1') \in R_1$. We define the team T_2' as follows:
  $T_2' = \{w_2' \in M_2 \mid w_2' \text{ has an } R_2\text{-predecessor in } T_2$
  $\qquad\qquad$ and there is $w_1 \in T_1, w_1' \in T_1'$ with $(w_1, w_1') \in R_1$ and $w_1' \rightleftharpoons_{k-1} w_w'.\}$
  By definition, T_2' only contains R_2-successors of worlds in T_2, and for each world w_2' in T_2', there is some $w_1' \in T_1'$ with $w_1' \rightleftharpoons_k w_2'$. To complete the proof, it therefore remains to show that:

  1. For each $w_2 \in T_2$, there is some R_2-successor w_2' of w_2 with $w_2' \in T_2$,
  2. for each $w_1' \in T_1'$, there is some $w_2' \in T_2'$ with $w_1' \rightleftharpoons_k w_2'$.

  We now prove these claims:

  1. Let $w_2 \in T_2$. Then there is some $w_1 \in T_1$ with $w_1 \rightleftharpoons_k w_2$. By choice of T_1', there is some $w_1' \in T_1'$ with $(w_1, w_1') \in R_1$. Since $w_1 \rightleftharpoons_k w_2$, there is an R_2-successor w_2' of T_2' with $w_1' \rightleftharpoons_{k-1} w_2'$. Hence by choice of T_2', we know that $w_2' \in T_2'$, and w_2' is an R_2-successor of w_2.
  2. Let $w_1' \in T_1'$. Then w_1' is the R_1-successor of some $w_1 \in T_1$. Since $T_1 \rightleftharpoons_k T_2$, there is some $w_2 \in T_2$ with $w_1 \rightleftharpoons_k w_2$. By choice of T_1', there is an R_1-successor w_1' of w_1

with $w_1' \in T_1'$. Since $w_1 \rightleftharpoons_k w_2$, there is some R_2-successor w_2' of w_2 with $w_1' \rightleftharpoons_{k-1} w_2'$. Hence, by choice of T_2', we have $w_2' \in T_2'$ as required.

◀

The expressive power of classical modal logic (i.e., without team semantics) can be characterized by bisimulations. In particular, for every pointed model (M, w), there is a modal formula of modal depth k that exactly characterizes (M, w) up to k-bisimulation.

In the following, we restrict ourselves to a finite set of propositional variables.

3   Main Result: Expressiveness of MTL

In this section, we study the expressive power of MTL. As usual, we measure the expressive power of a logic by the set of properties expressible in it.

▶ **Definition 3.1.** A *team property* is a class of pairs (M, T) where M is a Kripke model and $T \neq \emptyset$ a team of M. For an MTL-formula φ, we say that φ *expresses* the property $\{(M, T) \mid M, T \models \varphi\}$.

Note that most variants of modal dependence logic have the *empty team property*, i.e., for all $\varphi \in$ EMINCL and all Kripke structures M, we have $M, \emptyset \models \varphi$, which obviously does not hold for MTL. However, it immediately follows from the bisimulation invariance of MTL that for every MTL formula φ one of the two possibilities hold:

- For all Kripke structures M, $M, \emptyset \models \varphi$.
- For all Kripke structures M, $M, \emptyset \not\models \varphi$.

For this reason we exclude the empty team in the statement of our results below, but we note that by the remarks above all results cover also the empty team.

▶ **Definition 3.2.** Let P be a team property. Then P is *invariant under k-bisimulation* if for each pair of Kripke models M_1 and M_2 and teams T_1 and T_2 with $(M_1, T_1) \rightleftharpoons_k (M_2, T_2)$ and $(M_1, T_1) \in P$, it follows that $(M_2, T_2) \in P$.

We introduce some (standard) notation. In a model M, the *distance* between two worlds w_1 and w_2 of M is the length of a shortest path from w_1 to w_2 (the distance is infinite if there is no such path). For a world w of a model M and a natural number d, the *d-neighborhood of w in M*, denoted $\mathrm{N}_M^d(w)$, is the set of all worlds w' of M such that the distance from w to w' is at most d. For a team T, with $\mathrm{N}_M^d(T)$ we denote the set $\cup_{w \in T} \mathrm{N}_M^d(w)$. We often identify $\mathrm{N}_M^d(T)$ and the model obtained from M by restriction to the worlds in $\mathrm{N}_M^d(T)$.

▶ **Definition 3.3.** A team property P is *d-local* for some $d \in \mathbb{N}$ if for all models M and teams T, we have

$$(M, T) \in P \text{ if and only if } (\mathrm{N}_M^d(T), T) \in P.$$

We say that P is *local*, if P is d-local for some $d \in \mathbb{N}$.

Since our main result establishes a connection between team properties definable in MTL and team properties definable in first-order logic, we also define what it means for a team property to be expressed by a first-order formula. For a finite set of propositional variables X, we define σ_X as the first-order signature containing a binary relational symbol E (for the edges in our model), a unary relational symbol T (for representing a team), and, for each variable $x \in X$, a unary relational symbol W_x (representing the worlds in which x is true). Kripke models M with teams T (where we only consider variables in X) directly

correspond to σ_X structures: A model $M = (W, R, \pi)$ and a team T uniquely determines the σ_X-structure $\mathcal{M}_{M,T}^{\mathrm{FO}}$ with universe W and the obvious interpretations of the symbols in σ_X.

We therefore say that a first-order formula φ over the signature σ_X expresses a team property P, if for all models M with a team T, we have that $(M, T) \in P$ if and only if $\mathcal{M}_{M,T}^{\mathrm{FO}} \models \varphi$. We can now state the main result of this paper:

▶ **Theorem 3.4.** *Let P be a team property. Then the following are equivalent:*

(i) *There is an* MTL*-formula which expresses P,*
(ii) *there is a first-order formula which expresses P and P is bisimulation-invariant,*
(iii) *P is invariant under k-bisimulation for some k,*
(iv) *P is bisimulation-invariant and local.*

This result characterizes the expressive power of MTL in several ways. The equivalence of points i and ii is a natural analog to the classic van Benthem theorem which states that standard modal logic directly corresponds to the bisimulation-invariant fragment of first-order logic. It is easy to see that characterizations corresponding to items iii and iv also hold in the classical setting. Our result therefore shows that MTL plays the same role for team-based modal logics as ML does for standard modal logic.

The connection between our result and van Benthem's Theorem [24, 25] is also worth discussing. Essentially, van Benthem's Theorem is the same result as ours, where "MTL" is replaced by "ML" and properties of pointed models (i.e., singleton teams) are considered. In ML, classical negation is of course freely available; however the property of a team being a singleton is clearly not invariant under bisimulation—but the property of a team having only one element *up to bisimulation* is. It therefore follows that each property of singleton teams that is invariant under bisimulation and that can be expressed in MTL can already be expressed in ML.

The remainder of Section 3 is devoted to the proof of Theorem 3.4. The proof relies on various formulas that characterize pointed models, teams of pointed models, or team properties up to k-bisimulation, for some $k \in \mathbb{N}$. In Table 1, we summarize the notation used in the following and explain the intuitive meaning of these formulas.

3.1 Expressing Properties in MTL and Hintikka Formulas

We start with a natural characterization of the semantics of splitjunction $\vee$ for ML-formulas.

▶ **Proposition 3.5.** *Let S be a non-empty finite set of* ML*-formulas, let M be a model and T a team. Then $M, T \models \bigvee_{\varphi \in S} \varphi$ if and only if for each world $w \in T$, there is a formula $\varphi \in S$ with $M, \{w\} \models \varphi$.*

Proof. Let M be a model and T a team. For $|S| = 1$, the claim follows from Proposition 2.2. Therefore assume that the lemma holds for $S' = \{\varphi_1, \ldots, \varphi_{n-1}\}$, and consider a set $S = S' \cup \{\varphi_n\}$. Then $\bigvee_{\varphi \in S} \varphi = \bigvee_{\varphi \in S'} \vee \varphi_n$.

First assume that $(M, T) \models \bigvee_{\varphi \in S}$. Then $(M, T) \models \bigvee_{\varphi \in S'} \varphi \vee \varphi_n$. Therefore, $T = T_1 \cup T_2$ with

- $M, T_1 \models \bigvee_{\varphi \in S'} \varphi$,
- $M, T_2 \models \varphi_n$.

Due to the induction assumption, we know that for every world $w \in T_1$, there is a formula $\varphi \in S' \subseteq S$ with $M, w \models \varphi$. Due to Proposition 2.2, we know that each world $w \in T_2$

Formula	Intuition	Defined in
$\phi_{M,w}^k$	Characterizes the pointed model (M,w) up to k-bisimilarity	Theorem 3.6
$\Phi^{\rightleftharpoons_k}$	All formulas of the form $\phi_{M,w}^k$ (this is a finite set)	Definition 3.7
$\Phi_{M,T}^{\rightleftharpoons_k}$	Formulas characterizing pointed models (M,w), where $w \in T$, up to k-bisimilarity (this is a finite set)	Definition 3.8
$\varphi_{M,T}^{\rightleftharpoons_k}$	Formula characterizing model M with T up to k-bisimilarity	Definition 3.9

■ **Table 1** Formulas and sets of formulas used in the proof of Theorem 3.4

satisfies the formula $\varphi_n \in S$, therefore each world $w \in T$ satisfies some formula $\varphi \in S$ as claimed.

For the converse, assume that for each $w \in T$, there is some formula $\varphi \in S$ with $M, w \models \varphi$. Let $T_1 = \{w \in T \mid M, w \models \varphi_i \text{ for some } i \leq n - 1\}$, and let $T_2 = \{w \in T \mid M, w \models \varphi_n\}$. By the prerequisites we know that $T = T_1 \cup T_2$. Due to induction, we have that $M, T_1 \models \bigvee_{\varphi \in S'} \varphi$, and, due to Proposition 2.2, we know that $M, T_2 \models \varphi_n$. It therefore follows that $M, T \models \bigvee_{\varphi \in S'} \varphi \vee \varphi_n = \bigvee_{\varphi \in S} \varphi$ as claimed. ◀

The following result is standard:

▶ **Theorem 3.6.** [**7, Theorem 32**]- *For each pointed Kripke model (M,w) and each natural number k, there is a* Hintikka formula *(or characteristic formula) $\phi_{M,w}^k \in \mathsf{ML}$ with $md(\phi_{M,w}^k) = k$ such that for each pointed model (M', w'), the following are equivalent:*

1. $M', w' \models \phi_{M,w}^k$,
2. $(M,w) \rightleftharpoons_k (M', w')$.

Clearly, we can choose the Hintikka formulas such that $\phi_{M,w}^k$ is uniquely determined by the bisimilarity type of (M,w). This implies that for k-bisimilar pointed models (M_1, w_1) and (M_2, w_2), the formulas ϕ_{M_1,w_1}^k and ϕ_{M_2,w_2}^k are identical.

It it clear that Theorem 3.6 does not hold for an infinite set of propositional symbols, since a finite formula can only specify the values of finitely many variables.

We now define the set of all Hintikka formulas that will appear in our later constructions. Informally, $\Phi^{\rightleftharpoons_k}$ is the set of all Hintikka formulas characterizing models up to k-bisimilarity:

▶ **Definition 3.7.** For $k \in \mathbb{N}$, the set $\Phi^{\rightleftharpoons_k}$ is defined as

$$\Phi^{\rightleftharpoons_k} = \left\{ \phi_{M,w}^k \mid (M,w) \text{ is a pointed Kripke model} \right\}.$$

An important observation is that $\Phi^{\rightleftharpoons_k}$ is a finite set: This follows since above, we chose the representatives $\phi_{M,w}^k$ to be identical for k-bisimilar pointed models, and since there are only finitely many pointed models up to k-bisimulation. Since $\Phi^{\rightleftharpoons_k}$ is finite, we can in the following freely use disjunctions over arbitrary subsets of $\Phi^{\rightleftharpoons_k}$ and still obtain a finite formula. We will make extensive use of this fact in the remainder of Section 3, often without reference.

Our next definition is used to characterize a team, again up to k-bisimulation. Since teams are sets of worlds, we use sets of formulas to characterize teams in the natural way, by choosing, for each world in the team, one formula that characterizes it.

▶ **Definition 3.8.** For a model M and a team T, let

$$\Phi_{M,T}^{\rightleftharpoons_k} = \{\varphi \in \Phi^{\rightleftharpoons_k} \mid \text{there is some } w \in T \text{ with } M, w \models \varphi\}.$$

Since $\Phi_{M,T}^{\rightleftharpoons k} \subseteq \Phi^{\rightleftharpoons k}$, it follows that $\Phi_{M,T}^{\rightleftharpoons k}$ is finite as well. In fact, it is easy to see that $\left|\Phi_{M,T}^{\rightleftharpoons k}\right|$ is exactly the number of k-bisimilarity types in T, i.e., the size of a maximal subset of T containing only worlds such that the resulting pointed models are pairwise non-k-bisimilar.

We now combine the formulas from $\Phi_{M,T}^{\rightleftharpoons k}$ to be able to characterize M and T (up to k-bisimulation) by a single formula:

▶ **Definition 3.9.** For a model M with a team $T \neq \emptyset$, let

$$\varphi_{M,T}^{\rightleftharpoons k} = \left(\bigwedge_{\varphi \in \Phi_{M,T}^{\rightleftharpoons k}} \mathsf{E}\varphi \right) \wedge \left(\bigvee_{\varphi \in \Phi_{M,T}^{\rightleftharpoons k}} \varphi \right).$$

Intuitively, the formula $\varphi_{M,T}^{\rightleftharpoons k}$ expresses that in a model M' and T' with $M', T' \models \varphi_{M,T}^{\rightleftharpoons k}$, for each world $w \in T$ there must be some $w' \in T'$ such that $(M, w) \rightleftharpoons_k (M', w')$, and conversely, for each $w' \in T'$, there must be some $w \in T$ with $(M, w) \rightleftharpoons_k (M', w')$, which then implies that (M, T) and (M', T') are indeed k-bisimilar.

From the above, it follows that $\varphi_{M,T}^{\rightleftharpoons k}$ is a finite MTL-formula. Therefore, with the above intuition, it follows that $\varphi_{M,T}^{\rightleftharpoons k}$ expresses k-bisimilarity with (M, T).

▶ **Proposition 3.10.** *Let M_1, M_2 be Kripke models with teams nonempty T_1, T_2. Then the following are equivalent:*

- $(M_1, T_1) \rightleftharpoons_k (M_2, T_2)$
- $M_1, T_1 \models \varphi_{M_2, T_2}^{\rightleftharpoons k}$.

Proof. First assume that $(M_1, T_1) \rightleftharpoons_k (M_2, T_2)$. To see that $M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}} \mathsf{E}\varphi$, let $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$. By definition, φ is an ML-formula with $md(\varphi) \leq k$ and there is some world $w_2 \in T_2$ such that $M_2, w_2 \models \varphi$. Due to the bisimulation condition, we know that there is a world $w_1 \in T_1$ with $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$. From Proposition 2.6, it follows that $M_1, w_1 \models \varphi$. Since $w_1 \in T_1$, it follows that $M_1, T_1 \models \mathsf{E}\varphi$. Since this is true for each $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$, it follows that $M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}} \mathsf{E}\varphi$.

It remains to show that $M_1, T_1 \models \bigvee_{\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}} \varphi$. Due to Lemma 3.5, it suffices to show that for each world $w_1 \in T_1$, there is some formula $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$ with $M_1, w_1 \models \varphi$. Therefore, let $w_1 \in T_1$. Due to the bisimulation condition, we know that there is some $w_2 \in T_2$ with $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$. Due to the definition of $\Phi_{M_2, T_2}^{\rightleftharpoons k}$, we know that for the formula $\varphi := \Phi_{M_2, w_2}^k$ we have that $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$ and $M_2, w_2 \models \varphi$. Since $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$, it follows that $M_1, w_1 \models \varphi$, and hence for each $w_1 \in T_1$ there is a formula φ as required.

For the converse, assume that $M_1, T_1 \models \varphi_{M_2, T_2}^{\rightleftharpoons k}$. In particular, it follows that $M_1, T_1 \models \bigvee_{\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}} \varphi$. Due to Lemma 3.5, it follows that for each $w_1 \in T_1$, there is some formula $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$ with $M_1, w_1 \models \varphi$. Due to the definition of $\Phi_{M_2, T_2}^{\rightleftharpoons k}$, we know that $\varphi \in \Phi^{\rightleftharpoons k}$, and that there is some $w_2 \in T_2$ with $M_2, w_2 \models \varphi$. From Theorem 3.6, it therefore follows that for each $w_1 \in T_1$, there is some $w_2 \in T_2$ with $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$.

Let w_2 be a world in T_2, and let $\varphi := \Phi_{M_2, w_2}^k$. It then follows that $\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}$. Since $M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M_2, T_2}^{\rightleftharpoons k}} \mathsf{E}\varphi$, it follows from the definition of the operator E that there is some $w_1 \in T_1$ with $M_1, w_1 \models \varphi$. Due to the choice of φ, Theorem 3.6 implies that $(M_1, w_1) \rightleftharpoons_k (M_2, w_2)$ as required.   ◀

3.2   Proof of Theorem 3.4

In this section, we prove our main result, Theorem 3.4.

3.2.1 Proof of equivalence 3.4.(i) $\leftrightarrow$ 3.4.(iii)

Proof. The direction i $\to$ iii follows immediately from Proposition 2.8. For the converse, assume that P is invariant under k-bisimulation. Without loss of generality assume $P \neq \emptyset$. We claim that the formula

$$\varphi_P := \mathbb{V}_{(M,T) \in P} \varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$$

expresses P.

First note that φ_P can be written as the disjunction of only finitely many formulas: Each $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$ is uniquely defined by a subset of the finite set $\Phi^{\overrightarrow{\rightleftharpoons}^k}$, therefore there are only finitely many formulas of the form $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$.

We now show that for each model M and team T, we have that $(M,T) \in P$ if and only if $M,T \models \varphi_P$. First assume that $(M,T) \in P$. Then the fact that $(M,w) \rightleftharpoons_k (M,w)$ for each model M, each world w and each number k and Proposition 3.10 imply that $M,T \models \varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$. Therefore, $M,T \models \varphi_P$. For the converse, assume that $M,T \models \varphi_P$. Then there is some $(M',T') \in P$ with $M,T \models \varphi_{M',T'}^{\overrightarrow{\rightleftharpoons}^k}$. Due to Proposition 3.10, it follows that $(M,T) \rightleftharpoons_k (M',T')$. Since P is invariant under k-bisimulation, it follows that $(M,T) \in P$ as required.   ◄

3.2.2 Proof of implication 3.4.(iii) $\to$ 3.4.(ii)

Proof. It suffices to show that P can be expressed in first-order logic. This follows using essentially the standard translation from modal into first-order logic. Since classical disjunction is of course available in first-order logic, the proof of the implication iii $\to$ i shows that it suffices to express each $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$ (expressing team-bisimilarity to M,T) in first-order logic.

Each of the Hintikka formulas $\phi_{M,w}^k$ (expressing bisimilarity to the pointed model M,w) is a standard modal formula, therefore an application of the standard translation gives a first-order formula $\phi_{M,w}^{k,\mathrm{FO}}$ with a free variable x such that for all models M' and worlds w', we have that $M',w' \models \phi_{M,w}^k$ if and only if $\mathcal{M}_{M',\emptyset}^{\mathrm{FO}} \models \phi_{M,w}^{k,\mathrm{FO}}(w)$. We now show how to express $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$ (expressing team-bisimilarity to M,T) in first-order logic.

Recall that $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$ is defined as $\left(\bigwedge_{\varphi \in \Phi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}} \mathsf{E}\varphi \right) \wedge \left(\bigvee_{\varphi \in \Phi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}} \varphi \right)$. Therefore, a first-order representation of $\varphi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}$ is given as

$$\left(\bigwedge_{\varphi \in \Phi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}} \exists w(T(w) \wedge \varphi^{\mathrm{FO}}(w)) \right) \wedge \left(\forall w \left(T(w) \implies \bigvee_{\varphi \in \Phi_{M,T}^{\overrightarrow{\rightleftharpoons}^k}} \varphi^{\mathrm{FO}}(w) \right) \right),$$

where φ^{FO} is the standard translation of φ into first-order logic as mentioned above. This concludes the proof.   ◄

3.2.3 Proof of implication 3.4.(ii) $\to$ 3.4.(iv)

Proof. Let φ be the first-order formula expressing P. Since φ is first-order, we know that φ is Hanf-local[1]. Let d be the Hanf-locality rank of φ. We show that φ is $2d$-local. Therefore, let M be a model with team T. We show that $\mathcal{M}_{M,T}^{\mathrm{FO}} \models \varphi$ if and only if $\mathcal{M}_{\mathrm{N}_M^{2d}(T),T}^{\mathrm{FO}} \models \varphi$. Since φ is bisimulation-invariant, it suffices to construct models M_1 and M_2 containing T such that

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[1] See Appendix A for a brief discussion of Hanf-locality

- (M_1, T) and (M, T) are team-bisimilar,
- (M_2, T) and $(\mathrm{N}_M^{2d}(T), T)$ are team-bisimilar,
- $\mathcal{M}_{M_1,T}^{\mathrm{FO}} \models \varphi$ if and only if $\mathcal{M}_{M_2,T}^{\mathrm{FO}} \models \varphi$.

We first define M^{DISS} as the model obtained from M by disconnecting $\mathrm{N}_M^{2d}(T)$ from the remainder of the model, i.e., by removing all edges between $\mathrm{N}_M^{2d}(T)$ and $M \setminus \mathrm{N}_M^{2d}(T)$. Since M^{DISS} is also obtained from $\mathrm{N}_M^{2d}(T)$ by adding the remainder of the model M without connecting the added worlds to $\mathrm{N}_M^{2d}(T)$, it is obvious that $(M^{\mathrm{DISS}}, T) \rightleftharpoons (\mathrm{N}_M^{2d}(T), T)$. We now define the models M_1 and M_2 such that $(M_1, T) \rightleftharpoons (M, T)$ and $(M_2, T) \rightleftharpoons (M^{\mathrm{DISS}}, T)$ (and hence $(M_2, T) \rightleftharpoons (\mathrm{N}_M^{2d}(T), T)$) as follows:

- M_1 and M_2 are obtained from M and M^{DISS} by adding the exact same components: For each $w \in M$ (note that M and M^{DISS} have the exact same set of worlds), countably infinitely many copies of $\mathrm{N}_M^{2d}(w)$ and of $\mathrm{N}_{M^{\mathrm{DISS}}}^{2d}(w)$ are added to both M_1 and M_2.
- for $n \in \mathbb{N}$, and $i \in \{1, 2\}$, with $C_{i,n}^{\mathrm{DISS}}(w)$, we denote the n-th copy of $\mathrm{N}_{M^{\mathrm{DISS}}}^{2d}(w)$ in M_i, the *center* of $C_{i,n}^{\mathrm{DISS}}(w)$ is the copy of w in $C_{i,n}^{\mathrm{DISS}}(w)$.
- for $n \in \mathbb{N}$, and $i \in \{1, 2\}$, with $C_{i,n}^{\mathrm{CONN}}(w)$, we denote the n-th copy of $\mathrm{N}_M^{2d}(w)$ in M_i, the *center* of $C_{i,n}^{\mathrm{CONN}}(w)$ is the copy of w in $C_{i,n}^{\mathrm{CONN}}(w)$.

In the above, when we "copy" a part of a (Kripke) model, this includes copying the values of the involved propositional variables in these worlds (this is reflected in the resulting first-order models in the obvious way). However, we stress that the team T is treated differently: The set T is not enlarged with the copy operation, i.e., a copy of a world in T is itself not an element of T.

Since M_1 and M_2 are obtained from M and M^{DISS} by adding new components that are not connected to the original models, it clearly follows that (M, T) and (M_1, T) are team-bisimilar, and (M^{DISS}, T) and (M_2, T) are team-bisimilar. Note that each w in the M-part of M_1 is the center of a $2d$-environment isomorphic to $C_{2,n}^{\mathrm{CONN}}(w)$, and each w in the M^{DISS}-part of M_2 is the center of a $2d$-environment isomorphis to $C_{1,n}^{\mathrm{DISS}}(w)$.

Since the models M (M^{DISS}) contain one copy of each $\mathrm{N}_M^{2d}(w)$ ($\mathrm{N}_{M^{\mathrm{DISS}}}^{2d}(w)$), both M_1 and M_2 contain countably infinitely many copies of each $\mathrm{N}_M^{2d}(w)$ and each $\mathrm{N}_{M^{\mathrm{DISS}}}^{2d}(w)$. Let S_1 be the subset of M_1 containing only the points from the M-part of M_1, plus the center of each $C_{1,n}^{\mathrm{CONN}}(w)$, and the center of each $C_{1,n}^{\mathrm{DISS}}(w)$. Similarly, let S_2 be the subset of M_2 containing only the points from the M^{DISS}-part of M_2 plus the centers of the added components.

Since M_1 and M_2 contain the same number of copies of each relevant neighborhood, there is a bijection $f \colon S_1 \to S_2$ such that for each $w \in S_1$, the $2d$-neighborhoods of w and $f(w)$ are isomorphic. Now f can be modified such that for each $w \in M$ which has distance at most d to a world in T, the value $f(w)$ is the corresponding world in the M^{DISS}-part of M_2. The thus-modified f now satisfies that for each $w \in S_1$, the d-neighborhoods of w and $f(w)$ are isomorphic. We can easily extend f to worlds in $C_{1,n}^{\mathrm{DISS}}$ and $C_{1,n}^{\mathrm{CONN}}$ that are not the center of their respective components by mapping such a world w in $C_{1,n}^{\mathrm{DISS}}$ to the copy of w in $C_{2,n}^{\mathrm{DISS}}$, and analogously for $C_{1,n}^{\mathrm{CONN}}$.

Therefore, we have constructed a bijection $f \colon M_1 \to M_2$ such that for each $w \in M_1$, the d-neighborhood of w in M_1 is isomorphic to the d-neighborhood of w in M_2. Since φ is Hanf-local with rank d, this implies that $\mathcal{M}_{M_1,T}^{\mathrm{FO}} \models \varphi$ if and only if $\mathcal{M}_{M_2,T}^{\mathrm{FO}} \models \varphi$, as required.   ◀

The proof of this implication uses ideas from Otto's proof of van Benthem's classical theorem presented in [18]. However our proof is based on the Hanf-locality of first-order

expressible properties, whereas Otto's proof uses Ehrenfeucht-Fraïssé games, as a consequence, our construction requires an infinite number of copies of each model due to cardinality reasons.

3.2.4 Proof of implication 3.4.(iv) → 3.4.(iii)

Proof. Assume that P is invariant under bisimulation, and P is k-local for some $k \in \mathbb{N}$. We show that P is invariant under k-bisimulation. Hence let $M_1, T_1 \rightleftharpoons_k M_2, T_2$. Since P is invariant under bisimulation, we can without loss of generality assume that M_1 and M_2 are directed forests, that M_1 contains only worlds connected to worlds in T_1, and analogously for M_2 and T_2. Since P is also k-local, we can also assume that M_1 contains no world with a distance of more than k to T_1, and analogously for M_2 and T_2. From these assumptions, it immediately follows that $M_1, T_1 \rightleftharpoons M_2, T_2$, and, since P is invariant under bisimulation, this implies that $(M_1, T_1) \in P$ if and only if (M_2, T_2) in P, as required.   ◄

4 Alternative logical characterisations for the bisimulation invariant properties

Research on variants of (modal) dependence logic has concentrated on logics defined in terms of independence and inclusion atoms. Analogously to MDL, these logics are invariant under bisimulation but are strictly less expressive than MTL [12]. On the other hand, extended modal dependence logic, EMDL, uses dependence atoms but allows them to be applied to ML-formulas instead of just proposition symbols [4]. This variant is also known to be a proper sub-logic of MTL being able to express all downwards-closed properties that are invariant under k-bisimulation for some $k \in \mathbb{N}$, and equivalent to ML($\oslash$) [11].

In this section we systematically study the expressive power of variants of EMDL replacing dependence atoms by independence and inclusion atoms. Depending on whether we also allow classical disjunction or not, this gives four logics, namely EMIL (Extended Modal Independence Logic), EMIL($\oslash$) (EMIL extended with classical disjunction), EMINCL (Extended Modal Inclusion Logic) and EMINCL($\oslash$) (EMINCL extended with classical disjunction). We study the expressiveness of these logics, and show that while EMINCL($\oslash$) is as expressive as MTL, for each of the other three logics there is an MTL-expressible property that cannot be expressed in the logic. In the last section, we also study the extension of ML by first-order definable generalised dependence atoms, and show that the resulting logic—even without the addition of classical disjunction—is equivalent to MTL.

4.1 Extended Modal Independence Logic (EMIL)

We first consider *Extended Modal Independence Logic* (EMIL). Syntactically, EMIL extends ML by the following: If $\overline{P}$, $\overline{Q}$, and $\overline{R}$ are finite sets of ML-formulas, then $\overline{P} \perp_{\overline{R}} \overline{Q}$ is an EMIL-formula. The semantics of this *extended independence atom* are defined by lifting the definition for propositional variables given in [12] to ML-formulas as follows.

For a formula φ and a world w, we write $\varphi(w)$ for the function defined as $\varphi(w) = 1$ if $M, \{w\} \models \varphi$, and $\varphi(w) = 0$ otherwise (the model M will always be clear from the context). For a set of formulas $\overline{F}$ and worlds w_1, w_2, we write $w_1 \equiv_{\overline{F}} w_2$ if $\varphi(w_1) = \varphi(w_2)$ for each $\varphi \in \overline{F}$.

$$M, T \models \overline{P} \perp_{\overline{R}} \overline{Q} \quad \Leftrightarrow \quad \forall w, w' \in T \colon w \equiv_{\overline{R}} w' \text{ implies } \exists w'' \in T \colon$$
$$w'' \equiv_{\overline{P}} w \text{ and } w'' \equiv_{\overline{Q}} w' \text{ and } w'' \equiv_{\overline{R}} w.$$

The extension of EMIL by classical disjunction $\oslash$ is denoted by EMIL($\oslash$).

We will next show that $\mathsf{EMIL}(\varovee)$ is a proper sub-logic of MTL. The following lemma will be used in the proof.

▶ **Lemma 4.1.** *Let $M = (W, R, \pi)$ be a Kripke model such that $R = \emptyset$ and $T \subseteq W$ a team. Then for all $\varphi \in \mathsf{EMIL}(\varovee)$ it holds that if $M, T \models \varphi$, then $M, \{w\} \models \varphi$ for all $w \in T$.*

Proof. A straight-forward induction on the construction of φ using the facts that a singleton team trivially satisfies all independence atoms, and the empty team satisfies all formulas of $\mathsf{EMIL}(\varovee)$. ◀

▶ **Theorem 4.2.** $\mathsf{EMDL} \subsetneq \mathsf{EMIL} \subseteq \mathsf{EMIL}(\varovee) \subsetneq \mathsf{MTL}$.

Proof. The first inclusion follows from the fact that dependence atoms can be expressed by independence atoms. The inclusion is strict since EMDL is downwards-closed and EMIL is not.

For the last inclusion, note that every property expressible in $\mathsf{EMIL}(\varovee)$ is invariant under bisimulation, hence it follows that MTL can express every $\mathsf{EMIL}(\varovee)$-expressible property due to Theorem 3.4. For the strictness, we show that there is a team property that is invariant under 0-bisimulation and which cannot be expressed in $\mathsf{EMIL}(\varovee)$. Let P be the property

$$P := \{(M, T) \mid M, T \models \mathsf{E}p\},$$

i.e., the class of (M, T) such that T contains at least one world in which p is satisfied. Consider the model M with worlds w_1 and w_2, where p is true in w_1 and false in w_2, and the accessability relation $R = \emptyset$. Let $T_1 = \{w_1, w_2\}$, and let $T_2 = \{w_2\}$. Obviously, $(M, T_1) \in P$ and $(M, T_2) \notin P$. By Lemma 4.1 for all $\mathsf{EMIL}(\varovee)$-formulas φ, if $M, T_1 \models \varphi$, then $M, T_2 \models \varphi$. This shows that there is no $\mathsf{EMIL}(\varovee)$-formula expressing P. ◀

4.2 Extended Modal Inclusion Logic

Analogously to EMIL, we now define *Extended Modal Inclusion Logic*, EMINCL. EMINCL extends the syntax of ML with the following rule: If $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_n$ are ML-formulas, then $(\varphi_1, \ldots, \varphi_n) \subseteq (\psi_1, \ldots, \psi_n)$ is an EMINCL-formula. The semantics of this *inclusion atom* are lifted from the first-order setting [5] to the extended modal case:

$M, T \models (\varphi_1, \ldots, \varphi_n) \subseteq (\psi_1, \ldots, \psi_n)$ if for every world $w \in T$ there is a world $w' \in T$ such that $\varphi_i(w) = \psi_i(w')$ for each $i \in \{1, \ldots, n\}$.

The extension of EMINCL by classical disjunction $\varovee$ is denoted by $\mathsf{EMINCL}(\varovee)$.

Analogously to first-order inclusion logic [6], the truth of EMINCL-formulas is preserved under unions of teams. Hence we get the following result.

▶ **Theorem 4.3.** EMINCL *is strictly less expressive than* MTL.

Proof. We show that there is a 0-bisimulation invariant property that cannot be expressed with EMINCL. For this, let P be the property

$$\{(M, T) \mid \text{there is exactly one } i \in \{1, 2\} \text{ with } M, T \models \mathsf{E}p_i\}.$$

Clearly (and also due to Theorem 3.4), P is invariant under 0-bisimulation. Now, let M be a model with worlds w_1 and w_2 such that in w_i, the variable p_i is true and p_{3-i} is false. Let $T_1 = \{w_1\}$, and $T_2 = \{w_2\}$. Then, by construction, $(M, T_1) \in P$ and $(M, T_2) \in P$, but $(M, T_1 \cup T_2) \notin P$. Now assume that φ is an EMINCL-formula that expresses P.

Then, in particular $M, T_1 \models \varphi$, $M, T_2 \models \varphi$ and $M, (T_1 \cup T_2) \not\models \varphi$. However, it easily follows that EMINCL is *union*-closed, i.e. if $M, T_1 \models \varphi$ and $M, T_2 \models \varphi$, then also $M, (T_1 \cup T_2) \models \varphi$ (see, e.g., [6], the property trivially transfers to the modal setting). Therefore, we have a contradiction. ◀

Next we want to show that EMINCL($\oslash$) is as powerful as MTL.

▶ **Theorem 4.4.** *Let P be a team property. Then the following are equivalent:*

1. *P is invariant under k-bisimulation.*
2. *There is an EMINCL($\oslash$)-formula φ with $md(\varphi) = k$ that characterizes P.*

Proof. The direction from 2 to 1 follows by a straight-forward extension of the proof of Proposition 2.8. For the converse, assume that P is invariant under k-bisimulation. From the proof of Theorem 3.4, we know that it suffices to construct an EMINCL($\oslash$)-formula φ that is equivalent to the MTL-formula $\oslash_{(M,T) \in P} \varphi_{M,T}^{\rightleftharpoons k}$. Since the $\oslash$-operator is available in EMINCL($\oslash$), it suffices to show how to express the formula $\varphi_{M,T}^{\rightleftharpoons k}$ for each model M and team T as an EMINCL($\oslash$)-formula. Recall that

$$\varphi_{M,T}^{\rightleftharpoons k} = \left(\bigwedge_{\varphi \in \Phi_{M,T}^{\rightleftharpoons k}} \mathsf{E}\varphi \right) \wedge \left(\bigvee_{\varphi \in \Phi_{M,T}^{\rightleftharpoons k}} \varphi \right).$$

The second conjunct already is an EMINCL($\oslash$)-formula, hence it suffices to show how $\mathsf{E}\varphi$ can be expressed for an ML-formula φ. As discussed earlier, $M, T \models \mathsf{E}\varphi$ for an ML-formula φ if and only if there is a world $w \in T$ with $M, \{w\} \models \varphi$. Hence from the semantics of the inclusion atom, it is clear that $\mathsf{E}\varphi$ is equivalent to $(x \vee \neg x) \subseteq (\varphi)$. This concludes the proof. ◀

4.3 ML **with FO-definable generalized dependence atoms**

In this section we show that MTL, and the bisimulation invariant properties, can be captured as the extension of ML by all generalized dependence atoms definable in first-order logic without identity. The notion of a generalized dependence atom in the modal context was introduced in [12]. A closely related notion was introduced and studied in the first-order context in [13]. The semantics of a generalized dependence atom D is determined essentially by a property of teams.

In the following we are interested in generalized dependence atoms definable by first-order formulae, defined as follows: Suppose that D is an atom of width n, that is, an atom that applies to n propositional variables (for example the atom in (2)). We say that D is FO-definable if there exists a FO-sentence ϕ over signature $\langle A_1, \ldots, A_n \rangle$ such that for all Kripke models $M = (W, R, \pi)$ and teams T,

$$M, T \models D(p_1, \ldots, p_n) \iff \mathcal{A} \models \phi,$$

where $\mathcal{A}$ is the first-order structure with universe T and relations $A_i^{\mathcal{A}}$ for $1 \leq i \leq n$, where for all $w \in T$, $w \in A_i^{\mathcal{A}} \Leftrightarrow p_i \in \pi(w)$.

In our "extended" setting the arguments to a generalized dependence atom $D(\varphi_1, \ldots, \varphi_n)$ can be arbitrary ML-formulas instead of propositional variables. Hence the relation A_i is now interpreted by the worlds of T in which φ_i is satisfied. We denote by ML$^{\mathsf{FO}}$ the extension of ML by all generalized dependence atoms D that are FO-definable without identity.

▶ **Theorem 4.5.** $\mathsf{ML}^{\mathsf{FO}}$ *is equally expressive as* MTL.

Proof. In the proof of Theorem 6.8 in [12] it is showed that $\mathsf{ML}^{\mathsf{FO}}$ is invariant under bisimulation in the case where generalised atoms may be applied only to propositional variables. The proof easily extends to the setting where arbitrary ML-formulas may appear as arguments to a generalised dependence atom. Therefore, $\mathsf{ML}^{\mathsf{FO}}$ is not more expressive than MTL. For the converse, let P be a property that can be expressed in MTL. From Theorem 3.4, it follows that P is invariant under k-bisimulation, and from the proof of Theorem 3.4, we know that it suffices to express the formula $\varovee_{(M,T)\in P}\varphi_{M,T}^{\rightleftharpoons_k}$ in $\mathsf{ML}^{\mathsf{FO}}$. We can do this with the following first-order definable atom (by suitably choosing the parameters $n, m \in \mathbb{N}$):

$M, T \models D(\varphi_1^1, \ldots, \varphi_n^1, \varphi_1^2, \ldots, \varphi_n^2, \ldots, \varphi_1^m, \ldots, \varphi_n^m)$ if and only if there is some $k \in \{1, \ldots, m\}$ such that each $w \in T$ satisfies some φ_i^k, and for each $j \in \{1, \ldots, n\}$, there is some $w \in T$ that satisfies φ_j^k.

The atom D can now be FO-defined by replacing the exists/for all quantifiers on the indices with disjunctions/conjunctions:

$$\bigvee_{k\in\{1,\ldots,m\}} \left(\forall x \left(A_1^k(x) \vee \cdots \vee A_n^k(x) \right) \wedge \bigwedge_{j\in\{1,\ldots,n\}} (\exists x \ A_j^k(x)) \right)$$

Then, the atom D applied to the formulas in $\varphi_{M,T}^{\rightleftharpoons_k}$ for all $(M,T) \in P$ gives a formula expressing P. ◀

5   Conclusion

Our results show that, with respect to expressive power, modal team logic is a natural upper bound for all the logics studied so far in the area of modal team semantics. Overall, an interesting picture of the characterization of the expressiveness of modal logics in terms of bisimulation emerges: Let us say that "invariant under bounded bisimulation" means invariant under k-bisimulation for some finite k. Then we have the following hierarchy of logics:

- Due to van Benthem's theorem [25], ML can exactly express all properties of pointed models that are FO-definable and invariant under bisimulation.
- Due to [11], ML with team semantics and extended with classical disjunction $\varovee$ can exactly express all properties of teams that are invariant under bounded bisimulation and additionally downwards-closed.
- Our result shows that ML with team semantics and extended with classical negation $\sim$ can exactly express all properties of teams that are invariant under bounded bisimulation.

A number of open questions in the realm of modal logics with team semantics remain:

1. In the proof of Theorem 4.5, for each k, there is only a finite width of the D-operator above needed to express all properties that are invariant under k-bisimulation. However, the theorem leaves open the question whether there is a "natural" atom D or an atom with "restricted width" that gives the entire power of MTL.
2. Can we axiomatize MTL? Axiomatizability of sublogics of MTL has been studied, e.g., in [26] and [20].

3. While we mentioned a number of complexity results on modal dependence logic and some of its extensions, this issue remains unsettled for full MTL. In particular, what is the complexity of satisfiability and validity of MTL?

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A   Hanf-locality of first-order logic

In the proof of the implication Proof of implication ii → iv of Theorem 3.4, we used Hanf-locality of first-order formualas. In the following, we briefly introduce the relevant definitions and state the main result. Our presentation is based on [14, Section 4].

Since in this paper, we only consider properties of Kripke models, our first-oder models are defined over a relational signature that includes (at most) a single binary relation R representing the Kripke relation of our models, a unary relation T representing a team in a Kripke model, and for each propositional variable, a unary relation representing the worlds in which the variable is true. In the following, we assume a fixed finite set of variables.

For a world $w \in M$, and a natural number r, the *radius r ball* around w is the set of worlds w' such that there is an (undirected) path in M between w and w' with length at most r. The r-neighborhood of w in M, denoted with $N_r^M(w)$, is the model M restricted to the radius r ball around w, with an additional constant interpreted as w.

For two models M_1 and M_2, we write $M_1 \leftrightarrows d M_2$, if there is a bijection $f \colon W_1 \to W_2$ such that for every $c \in W_1$, $N_d^{M_1}(c) \cong N_d^{M_2}(f(c))$.

▶ **Definition 1.1.** A property P of Kripke models is *Hanf-local* if there is some $d \geq 0$ such that for every Kripke models M_1 and M_2 with $M_1 \leftrightarrows_d M_2$, we have that M_1 has property P if and only if M_2 has property P. The smallest number d for which this is true is the Hanf-locality rank of P.

In our proof, we use the property that first-order logic is Hanf-local:

▶ **Theorem 1.2.** *[10]. If P is a property of Kripke models that can be expressed in first-oder logic, then P is Hanf-local.*