Neutral helicity

From page 94 of these lecture notes we know that the normal $\mathbf{n}$ to the neutral tangent plane is given by

$$\begin{aligned} g^{-1}N^2\,\mathbf{n} \;&=\; -\rho^{-1}\nabla\rho + \kappa\nabla P \;=\; -\rho^{-1}\left(\nabla\rho - \nabla P/c^2\right) \\ &=\; \alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}. \end{aligned} \tag{3.11.1}$$

It is natural to think that all these little tangent planes would link up and form a well-defined surface, but this is not actually the case in the ocean. In order to understand why the ocean chooses to be so ornery [bad-tempered] we need to understand what property the normal $\mathbf{n}$ to a surface must fulfill in order that the surface exists. We will find that this property is that the scalar product of the normal of the surface $\mathbf{n}$ and the curl of $\mathbf{n}$ must be zero everywhere on the surface; that is $\mathbf{n}\cdot\nabla\times\mathbf{n}$ must be zero everywhere on the surface.

In general, for a surface to exist in (x,y,z) space there must be a function $\phi(x,y,z)$ that is constant on the surface and whose gradient $\nabla\phi$ is in the direction of the normal to the surface, $\mathbf{n}$. That is, there must be an integrating factor $b(x,y,z)$ such that $\nabla\phi = b\mathbf{n}$. Assuming now that the surface does exist, consider a line integral of $b\mathbf{n}$ along a closed curved path in the surface. Since the line element of the integration path is everywhere normal to $\mathbf{n}$, the closed line integral is zero, and by Stokes's theorem, the area integral of $\nabla\times(b\mathbf{n})$ must be zero over the area enclosed by the closed curved path. Since the area element of integration $d\mathbf{A}$ is in the direction $\mathbf{n}$, it is clear that $\nabla\times(b\mathbf{n})\cdot d\mathbf{A}$ is proportional to $\nabla\times(b\mathbf{n})\cdot\mathbf{n}$. The only way that this area integral can be guaranteed to be zero for all such closed paths is if the integrand is zero everywhere on the surface, that is, if $\nabla\times(b\mathbf{n})\cdot\mathbf{n} = (\nabla b\times\mathbf{n})\cdot\mathbf{n} + b(\nabla\times\mathbf{n})\cdot\mathbf{n} = 0$, that is, if $\mathbf{n}\cdot\nabla\times\mathbf{n} = 0$ at all locations on the surface.

For the case in hand, the normal to the neutral tangent plane is in the direction $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}$ and we define the neutral helicity H^{n} as the scalar product of $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}$ with its curl,

$$H^{\mathrm{n}} \;\equiv\; \left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}\right)\cdot\nabla\times\left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}\right). \tag{3.13.1}$$

Neutral tangent planes (which do exist) do not link up in space to form a well-defined neutral surface unless the neutral helicity H^{n} is everywhere zero on the surface.

Recognizing that both the thermal expansion coefficient and the saline contraction coefficient are functions of $(S_{\mathrm{A}},\Theta,p)$, neutral helicity H^{n} may be expressed as the following four expressions, all of which are proportional to the thermobaric coefficient T_{b}^{Θ} of the equation of state,

$$\begin{aligned} H^{\mathrm{n}} \;&=\; \beta^{\Theta}\,T_{\mathrm{b}}^{\Theta}\,\nabla P\cdot\nabla S_{\mathrm{A}}\times\nabla\Theta \\ &=\; P_z\,\beta^{\Theta}\,T_{\mathrm{b}}^{\Theta}\left(\nabla_p S_{\mathrm{A}}\times\nabla_p\Theta\right)\cdot\mathbf{k} \\ &=\; g^{-1}N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k} \\ &\approx\; g^{-1}N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_a P\times\nabla_a\Theta\right)\cdot\mathbf{k} \end{aligned} \tag{3.13.2}$$

where P_z is simply the vertical gradient of pressure ($\mathrm{Pa\ m^{-1}}$) and $\nabla_n\Theta$ and $\nabla_p\Theta$ are the two-dimensional gradients of Θ in the neural tangent plane and in the horizontal plane (actually the isobaric surface) respectively. The gradients $\nabla_a P$ and $\nabla_a\Theta$ are taken in an approximately neutral surface. Neutral helicity has units of $\mathrm{m^{-3}}$. Recall that the thermobaric coefficient is given by

$$T_{\mathrm{b}}^{\Theta} \;=\; \beta^{\Theta}\left(\alpha^{\Theta}/\beta^{\Theta}\right)_P \;=\; \alpha_P^{\Theta} - \left(\alpha^{\Theta}/\beta^{\Theta}\right)\beta_P^{\Theta}\,. \tag{3.8.2}$$

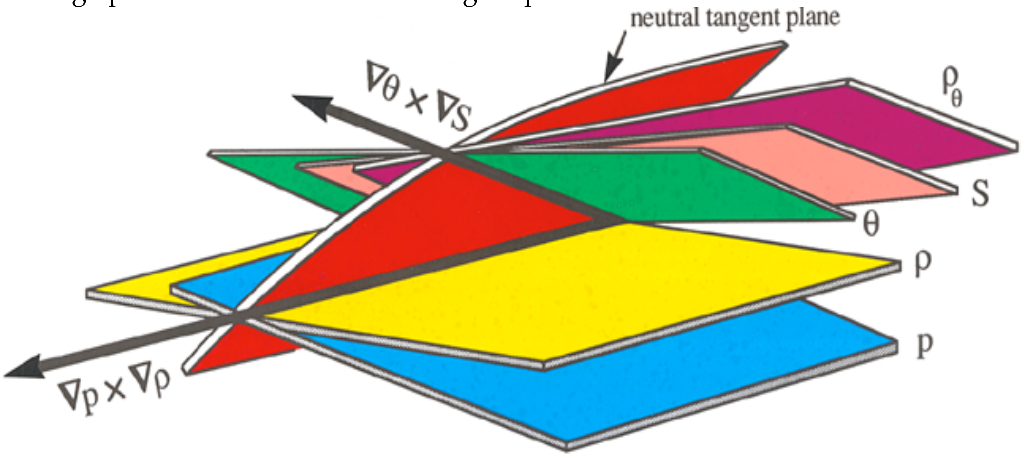
The geometrical interpretation of neutral helicity

How can we understand neutral helicity H^{n} geometrically? Recall the definition of a neutral tangent plane, Eqn. (3.11.2), namely

$$-\rho^{-1}\nabla_n\rho + \kappa\nabla_n P \;=\; \alpha^{\Theta}\nabla_n\Theta - \beta^{\Theta}\nabla_n S_{\mathrm{A}} \;=\; \mathbf{0}\,. \qquad (3.11.2)$$

This implies that the two lines $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ both lie in the neutral tangent plane. This is because along the line $\nabla P\times\nabla\rho$ both pressure and *in situ* density are constant, and along this line the neutral property is satisfied. Similarly, along the line $\nabla\Theta\times\nabla S_{\mathrm{A}}$ both Conservative Temperature and Absolute Salinity are constant, which certainly describes a line in the neutral tangent plane. Hence the picture emerges below of the geometry in (x,y,z) space of six planes, intersecting in one of the two lines $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$. The neutral tangent plane is the only plane that includes both of these desirable lines.

Why are these lines "desirable"? Well $\nabla P\times\nabla\rho$ is desirable because it is the direction of the "thermal wind", and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ is desirable because adiabatic and isohaline motion occurs along this line; a necessary attribute of a well-bred "mixing" plane such as the neutral tangent plane.



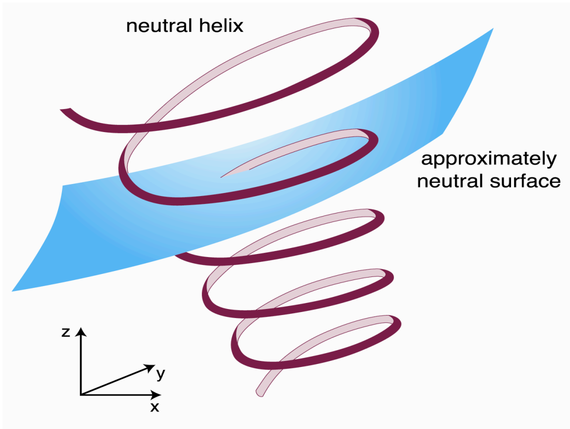
Prolonged gazing at the above figure while examining the definition of neutral helicity, H^{n}, Eqn. (3.13.2), shows that neutral helicity vanishes when the two vectors $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ coincide, and that this occurs when the two-dimensional gradients $\nabla_n\Theta$ are $\nabla_n P$ parallel.

Neutral helicity is proportional to the component of the vertical shear of the geostrophic velocity ($\mathbf{v}_z$, the "thermal wind") in the direction of the temperature gradient along the neutral tangent plane $\nabla_n\Theta$, since, from Eqn. (3.12.3) and the third line of (3.13.2) we find that

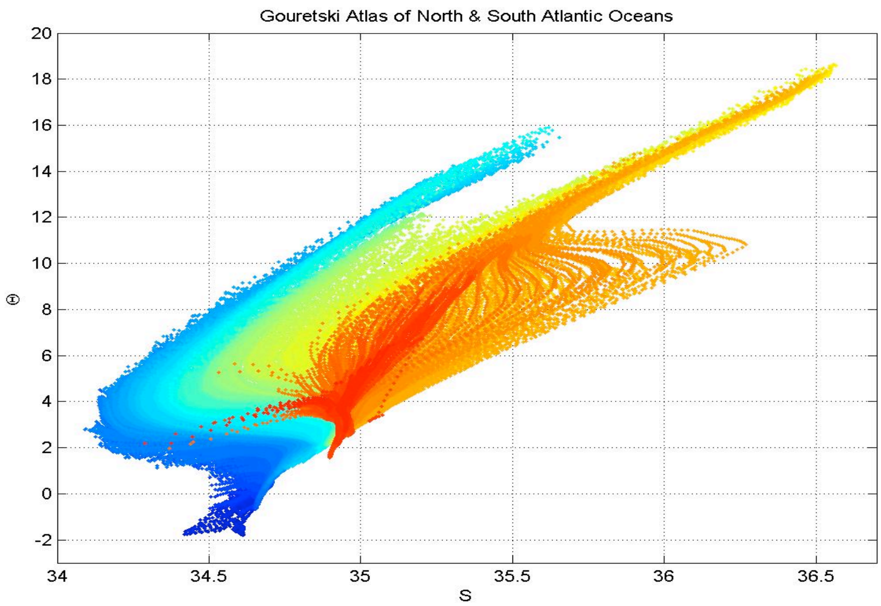
$$H^{\mathrm{n}} = \rho T_{\mathrm{b}}^{\Theta} f \mathbf{v}_z\cdot\nabla_n\Theta\,. \qquad (3.13.3)$$

Interestingly, for given magnitudes of the epineutral gradients of pressure and Conservative Temperature, neutral helicity is maximized when these gradients are perpendicular since neutral helicity is proportional to $T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k}$ (see Eqn. (3.13.2)), while the dianeutral advection of thermobaricity, $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$, is maximized when $\nabla_n\Theta$ and $\nabla_n P$ are parallel (see Eqn. **(A.22.4)**).

Because of the non-zero neutral helicity, H^{n}, in the ocean, lateral motion following neutral tangent planes has the character of helical motion. That is, if we ignore the effects of diapycnal mixing processes (as well as ignoring cabbeling and thermobaricity), the mean flow around ocean gyres still passes through any well-defined "density" surface because of the helical nature of neutral trajectories, caused in turn by the non-zero neutral helicity. We will return to this mean vertical motion caused by the ill-defined nature of "neutral surfaces" in a few pages.

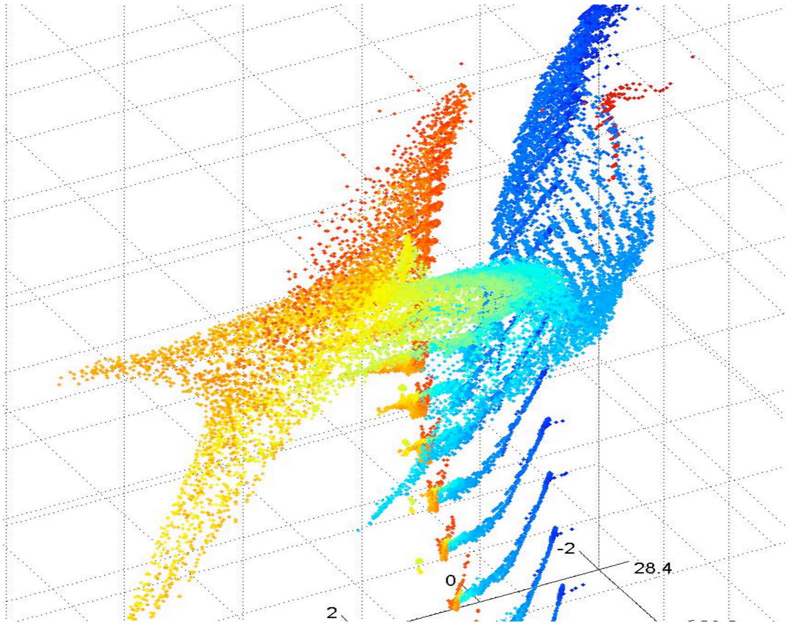


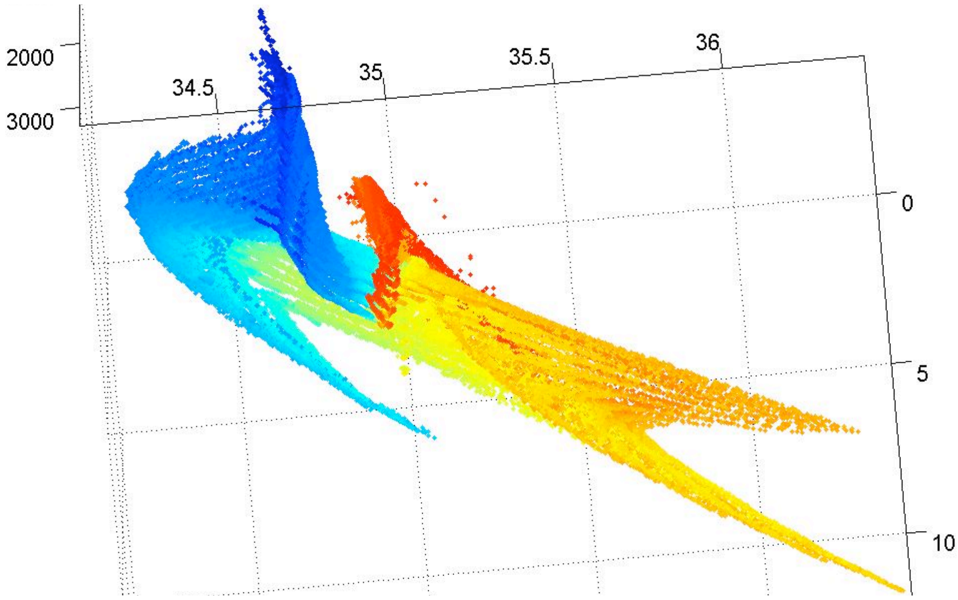
The skinny nature of the ocean; why is the ocean 95% empty?

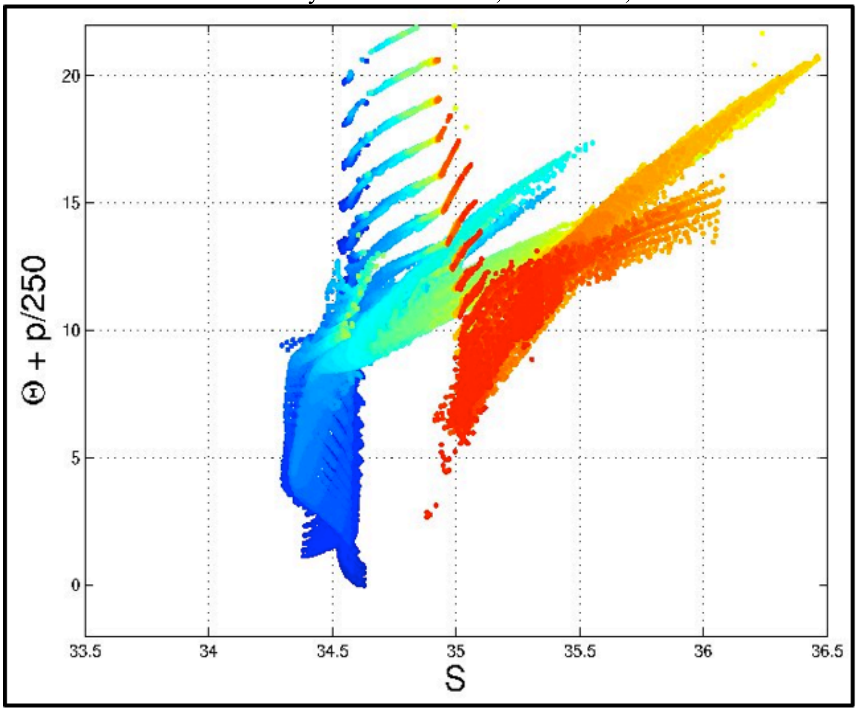


The above diagram contains all of the ocean hydrography below 200 dbar from both the North and South Atlantic ocean. The colour represents the latitude, with blue in the south, red in the north and green in the equatorial region. It is seen that the data fill the area on this $S_A - \Theta$ diagram, leaving no holes.

When considering the plotting of this same data on a three-dimensional $S_A - \Theta - p$ "plot", one could be forgiven for thinking that the data would fill in a solid shape in these three dimensions. But this is not observed. Rather than the $S_A - \Theta - p$ data occupying the volume inside, say, a packet of Toblerone chocolate, instead, the data resides on the cardboard of the Toblerone packet and the chocolate is missing.


-2
28.4
0
2


2000
3000
34.5
35
35.5
36
0
5
10


20
15
10
5
0
Θ + p/250
33.5
34
34.5
35
35.5
36
36.5
S

The skinny nature of the ocean; implication for neutral helicity

If all the (S_A, Θ, p) data from the whole global ocean were to lie exactly on a single surface in (S_A, Θ, p) space, we will prove that this requires $\nabla S_A \times \nabla\Theta \cdot \nabla P = 0$ everywhere in physical (x, y, z) space. That is, we will prove that the skinniness of the ocean hydrography in (S_A, Θ, p) space is a direct indication of the smallness of neutral helicity H^n.

Since, under our assumption, all the (S_A, Θ, p) data from the whole global ocean lies on the single surface in (S_A, Θ, p) space we have

$$f\left(S_A, \Theta, p\right) = 0 \qquad \text{(Twiggy\_01)}$$

for every (S_A, Θ, p) observation drawn for the whole global ocean in physical (x, y, z) space. Taking the spatial gradient of this equation in physical (x, y, z) space we have $\nabla f = 0$ since f is zero at every point in physical (x, y, z) space. Expanding ∇f in terms of the spatial gradients ∇S_A, $\nabla\Theta$, and ∇P, and taking the scalar product with $\nabla S_A \times \nabla\Theta$ we find that

$$\left.\frac{\partial f}{\partial P}\right|_{S_A,\Theta} \nabla P \cdot \nabla S_A \times \nabla\Theta = 0. \qquad \text{(Twiggy\_02)}$$

In the general case of $f_P \neq 0$, the result $\nabla P \cdot \nabla S_A \times \nabla\Theta = 0$ is proven. In the special case $f_P = 0$, f is independent of P so that we have a simpler equation for the surface f, being

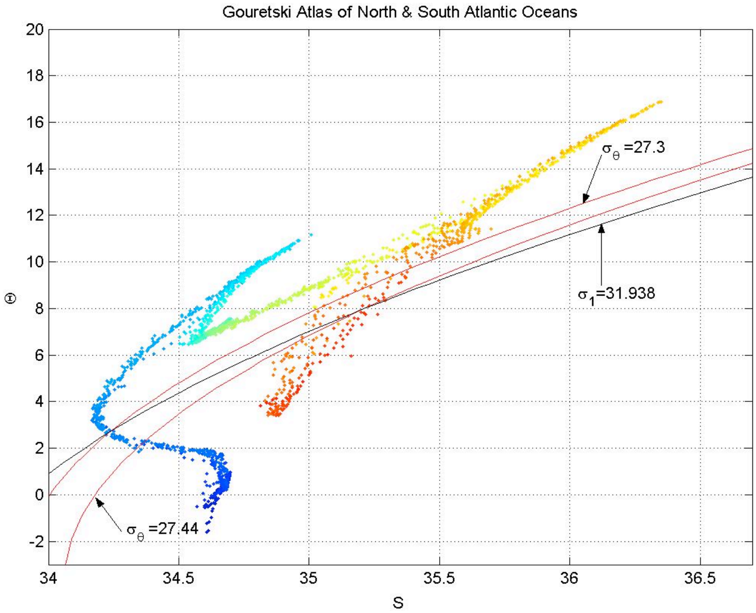
$$f\left(S_A, \Theta\right) = 0, \qquad \text{(Twiggy\_03)}$$

which is the equation for a single line on the (S_A, Θ) diagram; a single "water-mass" for the whole world ocean. In this case, changes in S_A are locally proportional to those of Θ so that $\nabla S_A \times \nabla\Theta = \mathbf{0}$ which also guarantees our required relation $\nabla P \cdot \nabla S_A \times \nabla\Theta = 0$.

Hence we have proven that the skinniness of the ocean hydrography in (S_A, Θ, p) space is a direct indication of the smallness of neutral helicity $H^n = \beta^\Theta T_b^\Theta \nabla P \cdot \nabla S_A \times \nabla\Theta$.

The skinny nature of the ocean; demonstrated from data at constant pressure

The diagram below is a cut at constant pressure through the above three-dimensional $S_A - \Theta - p$ data. The cut is at a pressure of 500 dbar. This diagram illustrates the smallness of neutral helicity from the perspective of the equation $H^n = P_z \beta^\Theta T_b^\Theta \left(\nabla_n S_A \times \nabla_n \Theta\right) \cdot \mathbf{k}$.



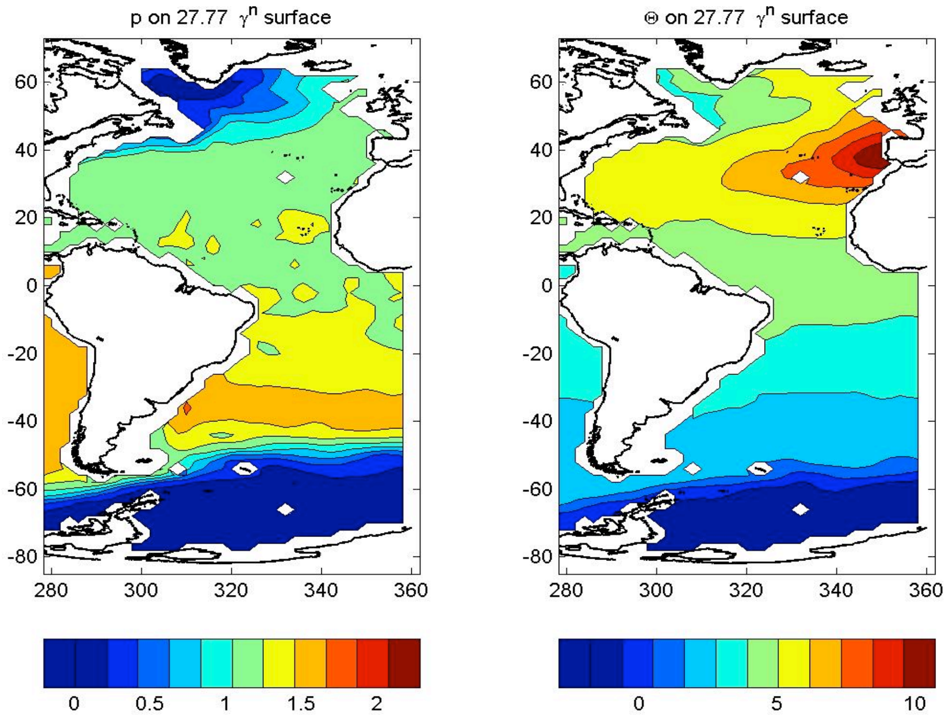
The skinny nature of the ocean; demonstrated from data on Neutral Density surfaces

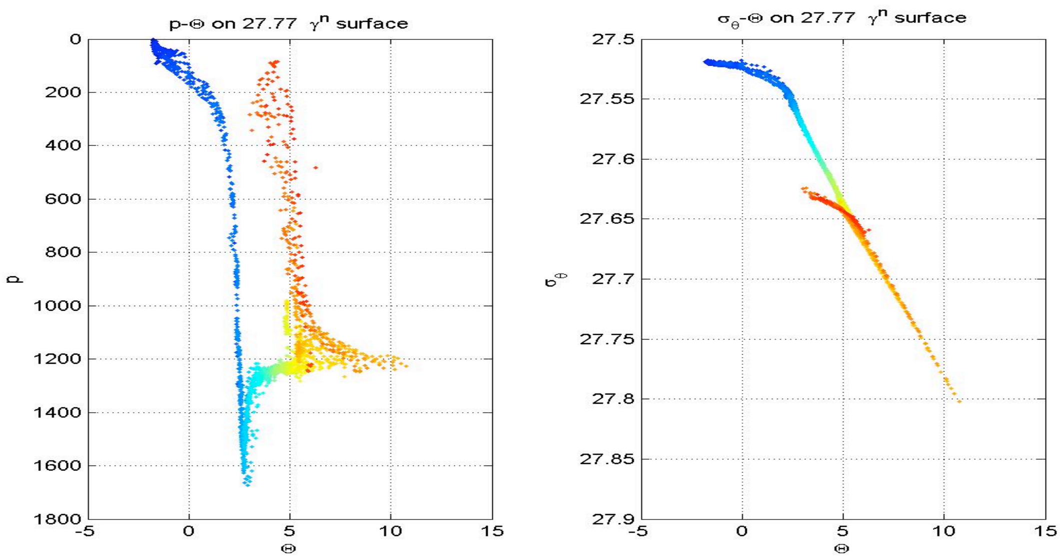
Here the "skinny" nature of the ocean will be demonstrated by looking at data on approximately neutral surfaces; Neutral Density γ^{n} surfaces. The following lines of the equation for neutral helicity

$$\begin{aligned} H^{\mathrm{n}} &= g^{-1} N^2 T_{\mathrm{b}}^{\Theta} \left(\nabla_n P \times \nabla_n \Theta \right) \cdot \mathbf{k} \\ &\approx g^{-1} N^2 T_{\mathrm{b}}^{\Theta} \left(\nabla_a P \times \nabla_a \Theta \right) \cdot \mathbf{k} \end{aligned} \qquad (3.13.2)$$

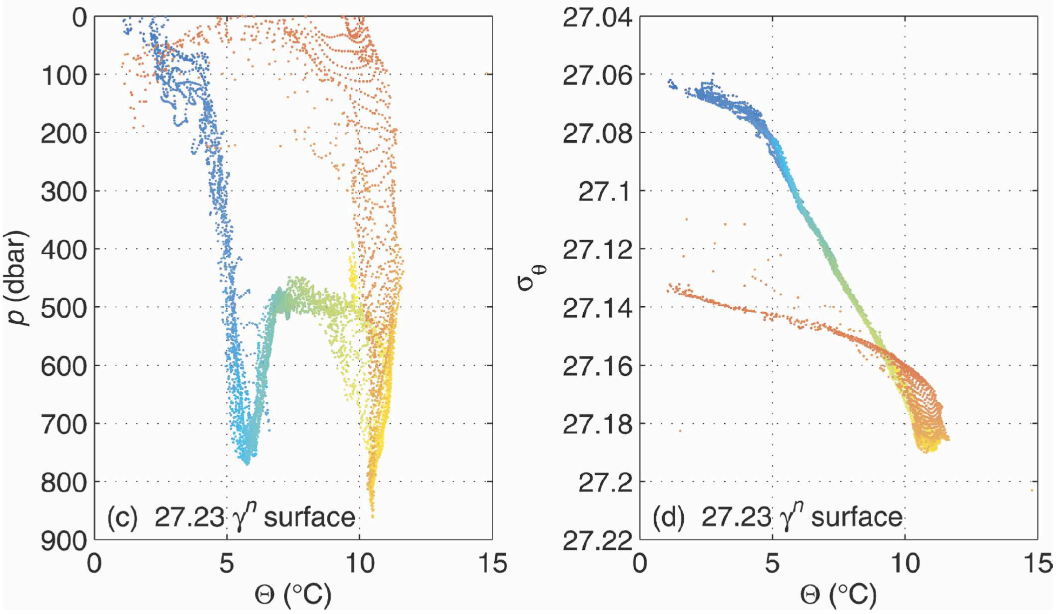
show that neutral helicity H^{n} will be small if the contours of P and of Θ on a γ^{n} surface are lined up; that is if $\nabla_a P$ and $\nabla_a \Theta$ are parallel.

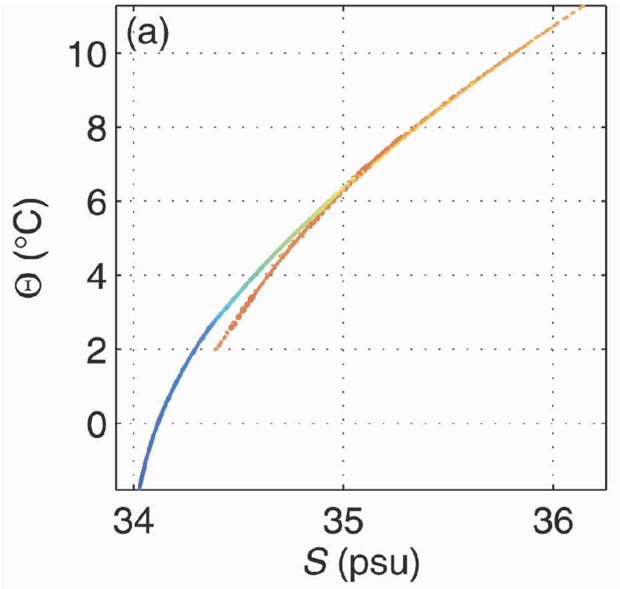
The ocean seems desperate to minimize H^{n}; either $\nabla_a P$ and $\nabla_a \Theta$ are parallel or where they are not parallel, one of $\nabla_a P$ or $\nabla_a \Theta$ is tiny.





Notice the rather large range of potential density of $0.28\,\mathrm{kg\,m^{-3}}$ on this Neutral Density surface. Also, the value of potential density at the northern hemisphere outcrop is larger than that at the southern hemisphere outcrop by about $0.1\,\mathrm{kg\,m^{-3}}$.





The above plots confirm that the ocean is rather "skinny" in $\left(S_{\mathrm{A}}, \Theta, p\right)$ space and hence that neutral helicity H^{n} is small in some sense (small compared to what?).

Note that while for some purposes a zero-neutral-helicity ocean,

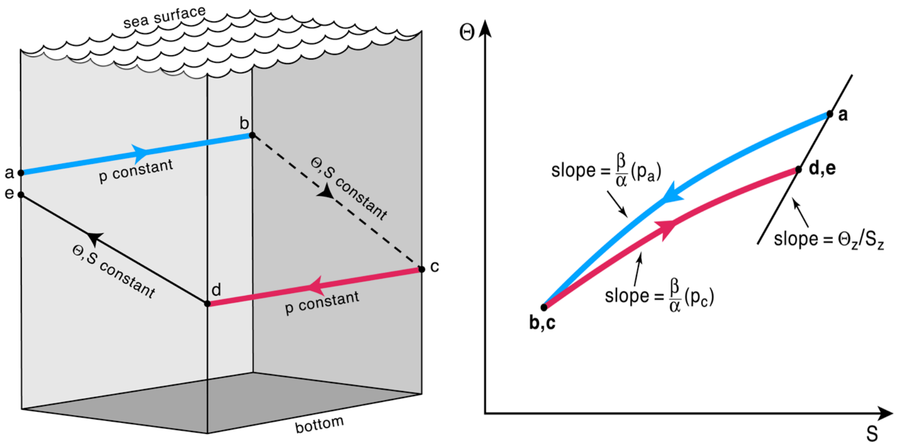
$$f\left(S_{\mathrm{A}}, \Theta, p\right) = 0 \qquad \text{(Twiggy\_01)}$$

might be a reasonable approximation, this $f\left(S_{\mathrm{A}}, \Theta, p\right) = 0$ surface is multi-valued along any particular axis. We saw this on the rotating view of the data in three $\left(S_{\mathrm{A}}, \Theta, p\right)$ dimensions. This multi-valued nature is also apparent on the last figure which is of only one approximately neutral surface. A slightly denser surface would have the same $\left(S_{\mathrm{A}}, \Theta\right)$ values in the Southern Atlantic as the above plot has in the North Atlantic.

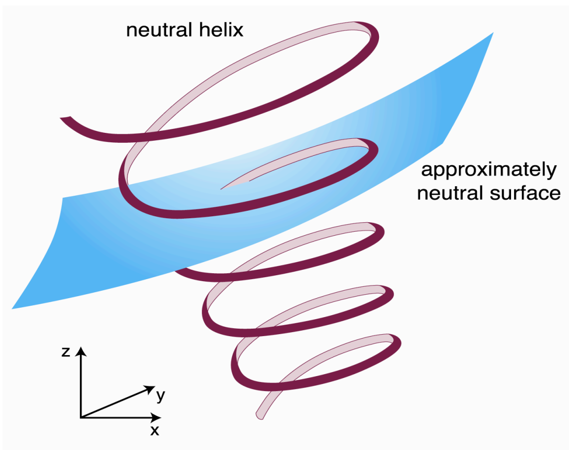
Note also in the above figures that where a particular Neutral Density surface comes to the surface (outcrops) in the North Atlantic, it has a greater potential density than in the Southern Ocean by between 0.07 kg m^{-3} and 0.14 kg m^{-3}. This is a general feature of the ocean; approximately neutral surfaces have different potential densities even at the reference pressure of that potential density. The northern hemisphere and southern hemisphere parts of a single ocean are separate branches in these multi-valued spaces.

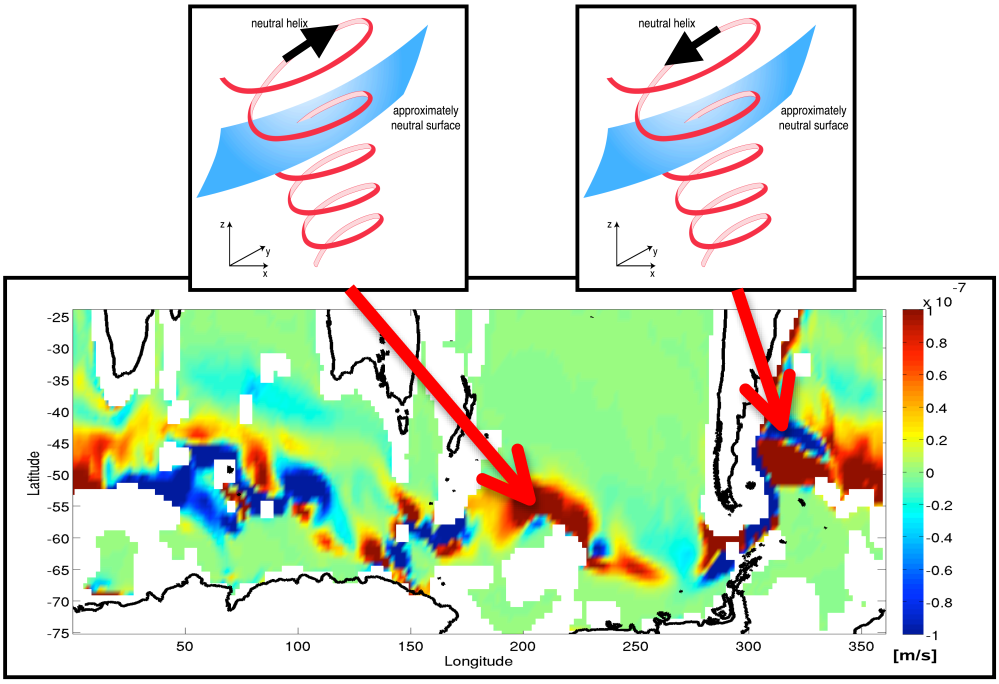
Consequences of non-zero neutral helicity

This diagram below is a simple example of the ill-defined nature of a "neutral surface" and the implication for mean dianeutral motion. The lateral mixing which causes the changes of S_A and Θ along this path occur at very different pressures. It is the rotation of the isopycnals on the $S_A - \Theta$ diagram (because of the different pressures) that causes the ill-defined nature of "neutral surfaces", that is, the helical nature of neutral trajectories. In this example $\nabla_a P$ and $\nabla_a \Theta$ are at right angles, that is, $\nabla_a P \cdot \nabla_a \Theta = 0$.



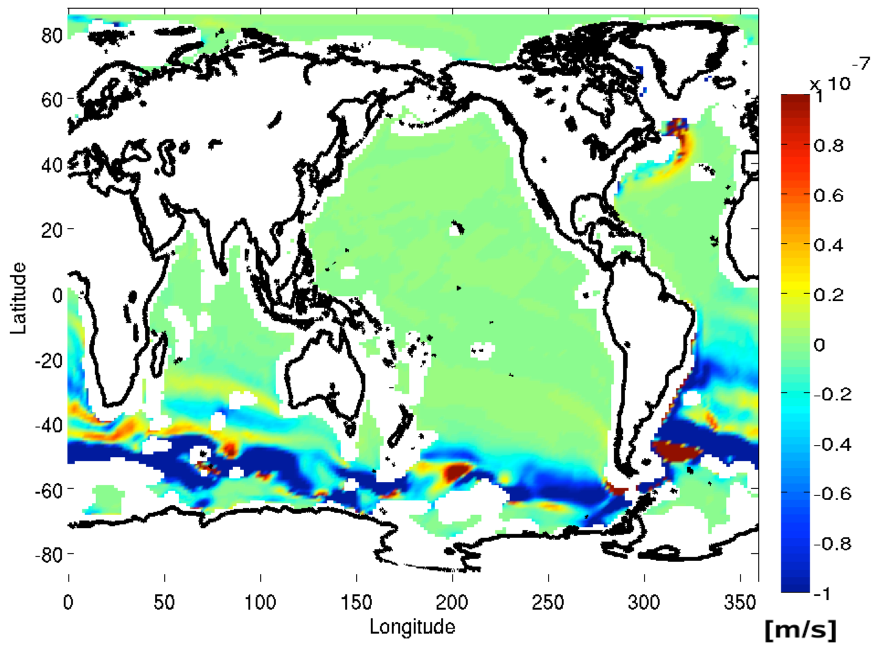
The cork-screwing motion as fluid flows along a helical neutral trajectory causes vertical dia-surface flow through any well-defined density surface. This mean diapycnal flow occurs in the absence of any vertical mixing process. That is, this mean vertical advection occurs in the absence of the dissipation of turbulent kinetic energy, and is additional to the other dianeutral advection processes, thermobaricity and cabbeling.

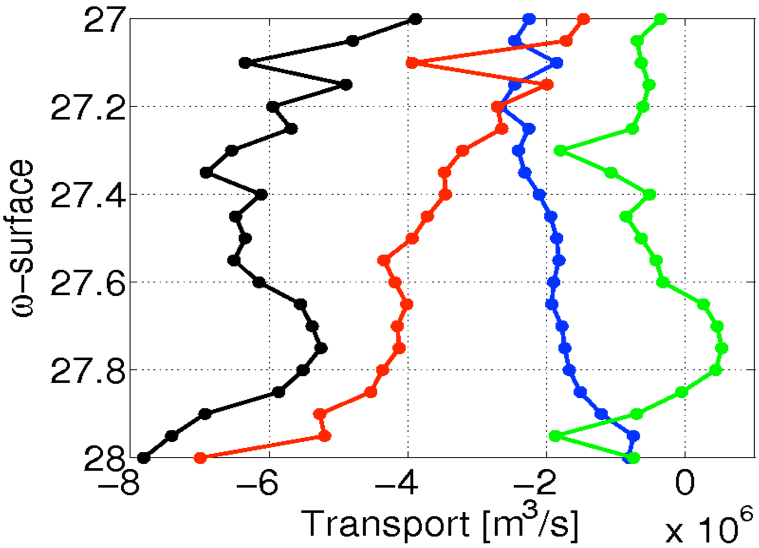




The figure above shows the vertical velocity through an approximately neutral surface caused by neutral helicity. That is, this is the actual vertical flow caused by the helical nature of neutral trajectories. The magnitude in the Southern Ocean is at leading order of $10^{-7}\ \mathrm{m\,s^{-1}}$, this being the canonical diapycnal velocity, dating back to Munk (1966).

The figure below is the total dianeutral velocity for all non-linear equation-of-state processes, namely thermobaricity, cabbeling and the helical nature of neutral trajectories.





When globally integrated over complete density surfaces, the total transport due to these non-linear processes can be calculated. In green is the mean dianeutral transport from the ill-defined nature of "neutral surfaces", blue is the dianeutral transport due to cabbeling, red due to thermobaricity, and black is the total global dianeutral transport due to the sum of these three non-linear processes.

We conclude from this that while the mean dianeutral transport from the ill-defined nature of "neutral surfaces" is of leading order locally, it spatially averages to a very small transport over a complete density surface. By contrast, cabbeling and thermobaricity are predominantly downwards advection everywhere, so there is little such cancellation on area integration with these processes.